Geometric Product for Multidimensional Dynamical Systems - Laplace Transform and Geometric Algebra

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***Abstract*—This contribution describes a new approach to a solution of multidimensional dynamical systems using the Laplace transform and geometrical product, i.e. using inner product (dot product, scalar product) and outer product (extended cross-product). It leads to a linear system of equations *Ax=0* or *Ax=b* which is equivalent to the outer product if the projective extension of the Euclidean system and the principle of duality are used. The paper explores property of the geometrical product in the frame of multidimensional dynamical systems.**

The proposed approach enables to avoid division operation and extents numerical precision as well. It also offers applications of matrix-vector and vector-vector operations in symbolic manipulation, which can lead to new algorithms and/or new formula. The proposed approach can be applied also for stability evaluation of dynamical systems. In the case of numerical computation, it supports vector operation and SSE instructions or GPU can be used efficiently.

Keywords*—*Linear system of equations, linear system of differential equations, Laplace transform, extended cross product, outer product, homogeneous coordinates, duality, geometrical algebra, dynamic systems, stability, GPGPU computation, SSE instructions.

I LAPLACE TRANSFORM

Integral transform maps a problem from the original domain to another one, where the problem can be solved in simple way and the result is converted back to the original domain using inverse transform. One such transform was discovered by Pierre-Simon Laplace in 1785, which is called the Laplace transform, now. It is an integral transform applied on a real function $f(t)$ with a real positive argument $t \geq 0$ and converts the function it to a complex function $F(s)$ with a complex argument $s = \delta + i\omega$.

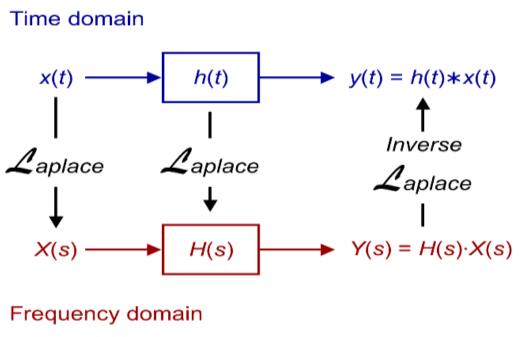

Fig.1.Laplace transform (taken from https://en.wikibooks.org [24])

The Laplace transform is defined as:

$$\mathcal{L}\{f(t)\} = F(s) \qquad F(s) = \int_0^{\infty} f(t)e^{-st}dt \tag{1}$$

The Laplace transform, see Fig.1, is often used for transform of differential system of equations to algebraic equations and convolution to multiplication [3],[4],[21],[25].

TABLE I. TYPICAL LAPLACE TRANSFORM PATTERNS

Time domain	s domain
$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
t	$1/_{s^2}$
$f(0)$	$\lim_{s\to\infty} sF(s)$
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
$f(t) * g(t)$ (convolution)	$F(s)G(s)$

It means, that a system of differential equations is transformed to a system of linear equations, which is to be solved and then this solution is transformed back to the time domain using inverse Laplace transform; in many cases the result is decomposed to some "patterns" for which the inverse transform is known.

The solution is then transformed back to the time domain using the inverse Laplace transform.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}dt \tag{2}$$

where α is taken so that all singularities of $F(s)$ are on the left of $Re(s)$. In many cases the result is decomposed to some "patterns" for which the inverse transform is known.

In the following, we introduce basic information projective representation, duality and geometric algebra.

II PROJECTIVE SPACE AND HOMOGENEOUS COORDINATES

The Euclidean space is used nearly exclusively in computational sciences. In some applications, like computer vision, computer graphics etc., the projective extension of the Euclidean space is used [2][9][20]. The projective extension in E^2 is defined as

$$X = \frac{x}{w} \qquad Y = \frac{y}{w} \qquad w \neq 0 \tag{3}$$

where x, y, w are homogeneous coordinates, i.e. $\boldsymbol{x} = [x, y: w]^T \in P^2$, $\boldsymbol{X} = (X, Y) \in E^2$ are coordinates in the Euclidean space. This concept is valid generally for the n-dimensional space. In general, a value in the projective space is represented as:

$$\boldsymbol{x} = [x_1, \dots, x_n: w]^T \quad \text{or} \quad \boldsymbol{x} = [x_0: x_1, \dots, x_n]^T \qquad \boldsymbol{x} \in P^n \tag{4}$$

where: x_0 stands for w; this notation is mostly used in mathematical resources. The symbol ":" means that the homogenous coordinate w is just a "scaling factor" and has no physical unit, while $x_1, \dots, x_n$ do have.

Let us introduce the extended cross product and its use with the projective space representation with simple geometrical examples for simplicity of explanation.

III DUALITY

The projective representation offers also one very important property – principle of duality. The principle of duality in E^2 states that any theorem remains true when we interchange the words "point" and "line", "lie on" and "pass through", "join" and "intersection", "collinear" and "concurrent" and so on. Once the theorem has been established, the dual theorem is obtained as described above [1][5][7]. In other words, the principle of duality says that in all theorems it is possible to substitute the term "point" by the term "line" and the term "line" by the term "point" etc. in and the given theorem stays valid. Similar duality is valid for E^3 as well, i.e. the terms "point" and "plane" are dual etc. it can be shown that operations "join" and "meet" are dual as well.

IV OUTER AND INNER PRODUCT

Solving system of linear algebraic equations is often used in many applications. However, methods for solution differ if the linear system of equations is homogeneous, i.e. $\boldsymbol{Ax} = \boldsymbol{0}$, or non-homogeneous $\boldsymbol{Ax} = \boldsymbol{b}$. If the projective extension of the Euclidean space is used and principle of duality applied, the both cases can be solved using extended cross-product as $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 \times \dots \times \boldsymbol{\alpha}_n$ or as $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \dots \wedge \boldsymbol{\alpha}_n$ if outer product is used, where $\boldsymbol{\alpha}_i$ is the $i-$th row of the matrix $\boldsymbol{A}$, resp. $[\boldsymbol{A}| - \boldsymbol{b}]$ [10]-[18].

In the case of differential equations, the Laplace transform transforms differential system to an algebraic system of equations. It can be seen that the extended cross-product does not use any division operation as would be expected in solution of a linear system of equations. In addition, it means that standard vector and/or matrix operations can be applied in further processing and solution of the system of equations can be avoided in principle. Symbolic manipulations using vector notation might lead to better understanding and possibly to derive new formulas.

The outer product (cross-product) of two vectors $\boldsymbol{a}, \boldsymbol{b}$ in E^3 is defined:

$$\boldsymbol{q} = \boldsymbol{a} \wedge \boldsymbol{b} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \tag{5}$$

where: $\boldsymbol{i} = [1,0,0]^T$, $\boldsymbol{j} = [0,1,0]^T$, $\boldsymbol{k} = [0,0,1]^T$ are unit vectors. The result of the cross-product $\boldsymbol{q}$ is a "bivector" which is an oriented area of a rhomboid in E^3 given by the vectors $\boldsymbol{a}, \boldsymbol{b}$. It should not be handled as a "movable vector" in general [17][18].

Let us consider computation of the intersection point $\boldsymbol{X} = (X, Y)$ of two given lines p_1 and p_2 in E^2:

$$p_1: a_1 X + b_1 Y + c_1 = 0 \qquad p_2: a_2 X + b_2 Y + c_2 = 0 \tag{6}$$

Multiplying those equations by $w \neq 0$ we get:

$$a_1 wX + b_1 wY + c_1 w = 0 \qquad a_2 wX + b_2 wY + c_2 w = 0 \tag{7}$$

Now, the projective representation can be used and as $x = wX$ and $y = wY$, i.e.:

$$\begin{aligned} a_1 wX + b_1 wY + c_1 w = a_1 x + b_1 y + c_1 w = 0 \\ a_2 wX + b_2 wY + c_2 w = a_2 x + b_2 y + c_2 w = 0 \end{aligned} \tag{8}$$

in the vector notation then:

$$\boldsymbol{p_1}^T \boldsymbol{x} = 0 \qquad \boldsymbol{p_2}^T \boldsymbol{x} = 0 \tag{9}$$

where $\boldsymbol{x} = [x, y: w]^T$ is the intersection point in the homogeneous coordinates of two lines $\boldsymbol{p}_1 = [a_1, b_1: c_1]^T$ and $\boldsymbol{p}_2 = [a_1, b_1: c_1]^T$.

It is easy to show that the intersection point $\boldsymbol{x}$ expressed in the projective space can be computed as [11][13]:

$$\boldsymbol{x} = \boldsymbol{p_1} \wedge \boldsymbol{p_2} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = [x, y: w]^T \tag{10}$$

where $\boldsymbol{i} = [1,0:0]^T$, $\boldsymbol{j} = [0,1:0]^T$, $\boldsymbol{k} = [0,0:1]^T$ are unit vectors in the projective space.

It is simple to prove that the above formula is correct. If two planes are parallel, then the coordinate $w = 0$, i.e. the intersection is in infinity.

The extended cross-product for E^4 has a form [17][18]:

$$\boldsymbol{q} = \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix} \tag{11}$$

where: $\boldsymbol{i} = [1,0,0:0]^T$, $\boldsymbol{j} = [0,1,0,0]^T$, $\boldsymbol{k} = [0,0,1:0]^T$, $\boldsymbol{l} = [0,0,0:1]^T$.

Now, due to the linearity it is possible to compute intersection of three planes $\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_3$ in P^3 as:

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2 \wedge \boldsymbol{\rho}_3 = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & \boldsymbol{l} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \tag{12}$$

where $\boldsymbol{\rho}_i = [a_i, b_i, c_i: d_i]^T$, i.e. $a_i X + b_i Y + c_i Z + d_i = 0$ and $\boldsymbol{x} = [x, y, z: w]^T$.

It means that we can solve $\boldsymbol{Ax} = \boldsymbol{b}$ using the extended cross-product. Now, we use the principle of duality for solving $\boldsymbol{Ax} = \boldsymbol{0}$ case.

It can be seen, that the implicit formulation and projective space representation offer clarity of formulation, simplicity and robustness of algorithms.

V GEOMETRIC ALGEBRA

The inner product is the most often algebraic construction in the n-dimensional Euclidean space. The Geometric Algebra (GA) is an inner product extension. The GA is not commutative and member of GA are called *multivectors.* The geometric product of two vectors in E^n is connected to the algebraic construction

$$\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v} \tag{13}$$

where $\boldsymbol{uv}$ is the geometric product, $\boldsymbol{u} \cdot \boldsymbol{v}$ is the inner product and $\boldsymbol{u} \wedge \boldsymbol{v}$ is the outer product (in E^3 equivalent to the cross product, i.e. $\boldsymbol{u} \times \boldsymbol{v}$). If $\boldsymbol{e}_i$ are orthonormal basis vectors, then

$$\begin{array}{ll} 1 & \text{0-vector (scalar)} \\ \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 & \text{1-vectors (vectors)} \\ \boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_3\boldsymbol{e}_1 & \text{2-vectors (bivectors)} \\ I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3 & \text{3-vector (pseudoscalar)} \end{array} \tag{14}$$

It can be easily proved that the inner product is

$$\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{2}(\boldsymbol{uv} + \boldsymbol{vu}) \tag{15}$$

There is something "strange" in the case of E^3 as the geometric product $\boldsymbol{uv} = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \wedge \boldsymbol{v}$ actually "accumulate" scalar value and result of the outer product, i.e. the cross product E^3, which is a *bivector*, actually not a vector. The size of it is an area of a rhomboid determined by the $\boldsymbol{u}, \boldsymbol{v}$ vectors the n-dimensional space in general. Due to the non-commutativity

$$\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u} \qquad \boldsymbol{uu} = \boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}|^2 \tag{16}$$

for all $\boldsymbol{u} \in R^n$. It means, that there is an inverse defined as

$$\boldsymbol{u}^{-1} = \boldsymbol{u} / |\boldsymbol{u}|^2 \tag{17}$$

There is another "object" called a *blade*. A *k-blade* $\boldsymbol{B}$ is a subspace given by orthogonal vectors $\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k}$, where $\boldsymbol{e}_i \neq \boldsymbol{e}_j$. Similar operations with vectors, operations with k*-blades* are introduced [2][5][6][19].

In the next, a modification of the geometric product for projective space is shortly described, as a user should be careful as the projective space is not just one dimension more in formulas.

VI GEOMETRIC PRODUCT AND PROJECTIVE SPACE

In geometry, scalar product (dot product), i.e. inner product, and cross product, i.e. outer product, are mostly used. However, there is no clear, simple geometric model, what the geometric product actually means, as the result of it is a set of objects with different properties and dimensionalities in the E^n case. Also computation of geometric product seems to be complicated even for the E^3 case and especially of homogeneous coordinates are to be used.

Geometric product $\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$ of two vectors, using homogeneous coordinates, as $\boldsymbol{a} = [a_1, a_2, a_3 : a_4]^T$ and $\boldsymbol{b} = [b_1, b_2, b_3 : b_4]^T$ can be easily computed using standard matrix operation, *respecting anti-commutativity*, as:

$$\begin{aligned} \boldsymbol{ab} \underset{repr}{\Longleftrightarrow} \boldsymbol{ab}^T &= \boldsymbol{a} \otimes \boldsymbol{b} \\ &= \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ b_1a_2 & a_2b_2 & a_2b_3 & a_2b_4 \\ b_1a_3 & b_2a_3 & a_3b_3 & a_3b_4 \\ b_1a_4 & b_2a_4 & b_3a_4 & a_4b_4 \end{bmatrix} \\ &= \boldsymbol{L} + \boldsymbol{U} + \boldsymbol{D} \end{aligned} \tag{18}$$

where $\boldsymbol{L}, \boldsymbol{U}, \boldsymbol{D}$ are **L**ower triangular, **U**pper triangular, **D**iagonal matrices, a_4, b_4 are the homogeneous coordinates.
Note, that the outer product is anti-commutative as $\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i$ for $i \neq j$.

It can be seen that the diagonal of the matrix $\boldsymbol{D}$ actually represents the inner product in the projective representation:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= [(a_1b_1 + a_2b_2 + a_3b_3) : a_4b_4]^T \\ &\triangleq \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_4b_4} \end{aligned} \tag{19}$$

where $\triangleq$ means projectively equivalent.

The outer product is then represented by due to anti-commutativity as

$$\begin{aligned} \boldsymbol{a} \wedge \boldsymbol{b} &\underset{repr}{\Longleftrightarrow} \sum_{\boldsymbol{i,j=1} \,\&\, \boldsymbol{i \neq j}}^{\boldsymbol{3}} a_i b_j \boldsymbol{e}_i \boldsymbol{e}_j \\ &= \sum_{\boldsymbol{i,j=1} \,\&\, \boldsymbol{i>j}}^{\boldsymbol{3}} \left(a_i b_j \boldsymbol{e}_i \boldsymbol{e}_j - b_i a_j \boldsymbol{e}_i \boldsymbol{e}_j\right) \\ &= \sum_{\boldsymbol{i,j} \,\&\, \boldsymbol{i>j}}^{\boldsymbol{3,3}} \left(a_i b_j - b_i a_j\right) \boldsymbol{e}_i \boldsymbol{e}_j \end{aligned} \tag{20}$$

and it can be seen a close relation to the Plücker coordinates as well. The outer product can be used for a solution of a linear system of equations, which is needed for a solution of multidimensional dynamical systems using the Laplace transform.

VII SOLUTION OF LINEAR SYSTEMS

The system $\boldsymbol{Ax} = \boldsymbol{b}$ can be rewritten as:

$$[\boldsymbol{A}| -\boldsymbol{b}] \begin{bmatrix} \boldsymbol{x} \\ w \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} & -b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & -b_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ w \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ 0 \end{bmatrix} \tag{21}$$

and solution is given using the extended cross-product as:

$$\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 \wedge \dots \wedge \boldsymbol{\alpha}_n = [x_1, \dots, x_n : w]^T \tag{22}$$

where $\boldsymbol{\alpha}_i = [a_{i1}, \dots, a_{in} : b_i]$, $i = 1, \dots, n$.

It should be noted, that the presented approach offers an unique approach to a solution of both types of the linear systems of equations, i.e. $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{Ax} = \boldsymbol{b}$. It also offers possibility of further symbolic manipulations using standard vector operations, including dot product and cross-product.

Now, it is possible to apply the above presented concept with the Laplace transform to a solution of the linear system of differential equations.

VIII OUTER PRODUCT AND LAPLACE TRANSFORM

Let us consider again a simple system of differential equations:

$$x' = 3x - 3y + 2 \qquad y' = -6x - t \tag{23}$$

with initial conditions $x(0) = 1, y(0) = -1$.

Applying the Laplace transform, we obtain a system of linear algebraic equations with respect to x, y as:

$$\begin{aligned} sX(s) - x(0) &= 3X(s) - 3Y(s) + \frac{2}{s} \\ sY(s) - y(0) &= -6X(s) - \frac{1}{s^2} \end{aligned} \tag{24}$$

Including initial conditions this yield to:

$$\begin{aligned} (s-3)X(s) + 3Y(s) &= 1 + \frac{2}{s} \\ 6X(s) + sY(s) &= -1 - \frac{1}{s^2} \end{aligned} \tag{25}$$

It means that the system described by equations:

$$\begin{bmatrix} s-3 & 3 \\ 6 & s \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s} \\ -\dfrac{s^2+1}{s^2} \end{bmatrix} \tag{26}$$

In the projective representation, it is represented as:

$$\begin{aligned} \bar{\boldsymbol{x}}(s) = \boldsymbol{\xi}_1(s) \wedge \boldsymbol{\xi}_2(s) &= \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ s-3 & 3 & -\dfrac{s+2}{s} \\ 6 & s & \dfrac{s^2+1}{s^2} \end{bmatrix} \\ &= [\bar{x}(s), \bar{y}(s) : \bar{w}(s)]^{\mathrm{T}} \end{aligned} \tag{27}$$

where:

$$\begin{aligned} \boldsymbol{\xi}_1(s) &= \left[s-3, 3 : -\frac{s+2}{s} \right]^T \\ \boldsymbol{\xi}_2(s) &= \left[6, s : \frac{s^2+1}{s^2} \right]^T \end{aligned} \tag{28}$$

Applying the extended cross-product, a solution is obtained:

$$\begin{aligned} \bar{\boldsymbol{x}}(s) &= [\bar{x}(s), \bar{y}(s) : \bar{w}(s)]^{\mathrm{T}} \\ &= \begin{bmatrix} 3\dfrac{s^2+1}{s^2} + \dfrac{s+2}{s} s \\ -6\dfrac{s+2}{s} - (s-3)\dfrac{s^2+1}{s^2} \\ s(s-3) - 18 \end{bmatrix} \end{aligned} \tag{29}$$

i.e.

$$\begin{aligned} \bar{x}(s) &= 3\frac{s^2+1}{s^2} + \frac{s+2}{s} s \\ \bar{y}(s) &= -6\frac{s+2}{s} - (s-3)\frac{s^2+1}{s^2} \\ \bar{w}(s) &= s(s-3) - 18 \end{aligned} \tag{30}$$

If the conversion to the Euclidean space representation is needed, then:

$$\begin{aligned} X(s) = \frac{\bar{x}(s)}{\bar{w}(s)} &= \frac{3\dfrac{s^2+1}{s^2} + s + 2}{s(s-3) - 18} \\ &= \frac{s^2(s+2) + 3s^2 + 3}{s^2(s^2 - 3s - 18)} \\ &= \frac{s^3 + 5s^2 + 3}{s^2(s^2 - 3s - 18)} \end{aligned} \tag{31}$$

and

$$\begin{aligned} Y(s) = \frac{\bar{y}(s)}{\bar{w}(s)} &= -\frac{(s-3)\dfrac{s^2+1}{s^2} + 6\dfrac{s+2}{s}}{s(s-3) - 18} \\ &= -\frac{s^3 + 3s^2 + 13s - 3}{s^2(s^2 - 3s - 18)} \end{aligned} \tag{32}$$

Of course, the above presented approach can be applied with including general (unspecified) initial conditions [22].

The presented approach demonstrates an equivalence of system of linear equations and the extended cross product, a more general approach can be found in [6][8][19][23]. It also enables symbolic manipulation and non-trivial transforms described in [14].

However, there is also inner product, which is part of the geometric product.

IX INNER PRODUCT AND LAPLACE TRANSFORM

Let us consider the recent example.

$$\boldsymbol{\xi}_1(s) = \left[s-3, 3 : -\frac{s+2}{s} \right]^T \qquad \boldsymbol{\xi}_2(s) = \left[6, s : \frac{s^2+1}{s^2} \right]^T \tag{33}$$

Then the inner product of $\boldsymbol{\xi}_1(s) \cdot \boldsymbol{\xi}_2(s)$ using the projective notation is

$$\begin{aligned} &\boldsymbol{\xi}_1(s) \cdot \boldsymbol{\xi}_2(s) \\ &= \left[(6(s-3) + 3\mathrm{s}) : \left(-\frac{s+2}{s} \right) \left(\frac{s^2+1}{s^2} \right) \right] \end{aligned} \tag{34}$$

It results using the inverse Laplace transform into:

$$\begin{aligned} &\mathcal{L}_s^{-1} \left[(6(s-3) + 3\mathrm{s}) : \left(-\frac{s+2}{s} \right) \left(\frac{s^2+1}{s^2} \right) \right] (t) \\ &= -\frac{288}{5} e^{-2t} - \frac{9}{5}(4 \sin t + 3 \cos t) + 36\, \delta(t) \\ &\qquad - 9\, \delta'(t) \end{aligned} \tag{35}$$

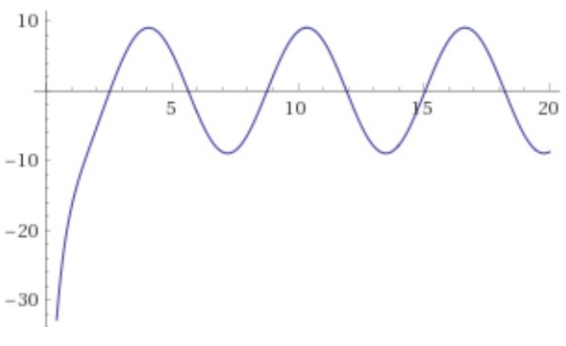

Fig.2. Result of the inner product in time (produced by WolframAlpha)

The question is what the inner product part in the geometric product does mean in the frame of the multidimensional dynamical system solution.

This is even more important question as the geometric algebra enables to manipulate with objects having different dimensionality efficiently keeping clarity and simplicity of formulation and finally robustness of the solution.

X CONCLUSION

The geometric product is a general tool enabling to describe multidimensional objects and used for a description of physical problems, including geometrical problems and their solutions as well. This paper describes application of the outer product, which is a part of the geometrical product, for the multidimensional dynamical systems. This led to a detection, that there is "hidden", resp. not used, part, i.e. influence of the inner product within the geometric product. In the example presented above, the inner product reflect some kind of oscillating behavior, which will be analyzed in future more detailed research.

The presented approach is easily applicable to multidimensional dynamical systems.

ACKNOWLEDGMENT

The author would like to thank to colleagues at the University of West Bohemia in Plzen for fruitful discussions and to anonymous reviewers for their comments and hints, which helped to improve the manuscript significantly.

The research was supported by projects Czech Science Foundation (GACR), No. GA17-05534S and partially by SGS 2016-013.

REFERENCES

[1] Coxeter,H.S.M.: Introduction to Geometry, J.Wiley, 1961.

[2] Dorst,L., Fontine,D., Mann,S.: Geometric Algebra for Computer Science, Morgan Kaufmann, 2007

[3] Duffy.D.G.: Transform Methods for Solving Partial Differential Equations, CRC Press, 1994

[4] Franklin,G., Powell,D., Emami-Naeini,A.:. Feedback Control of Dynamic Systems, Prentice-Hall, 2002.

[5] Gonzales Calvet,R.: Treatise of Plane Geometry through Geometric Algebra, 2007

[6] Hildenbrand,D.: Foundations of Geometric Algebra Computing, Springer Verlag, 2012

[7] Johnson,M.: Proof by Duality: or the Discovery of "New" Theorems, Mathematics Today, December 1996.

[8] Kanatani,K.: Undestanding geometric Algebra, CRC Press, 2015

[9] MacDonald,A.: Linear and Geometric Algebra, Charleston: Create Space, 2011

[10] Skala,V.: A New Approach to Line and Line Segment Clipping in Homogeneous Coordinates, The Visual Computer, Vol.21, No.11, pp.905-914, Springer Verlag, 2005

[11] Skala,V.: Length, Area and Volume Computation in Homogeneous Coordinates, International Journal of Image and Graphics, Vol.6., No.4, pp.625-639, 2006

[12] Skala,V.: Barycentric Coordinates Computation in Homogeneous Coordinates, Computers & Graphics, Elsevier, ISSN 0097-8493, Vol. 32, No.1, pp.120-127, 2008

[13] Skala,V.: Intersection Computation in Projective Space using Homogeneous Coordinates, Int.Journal on Image and Graphics, ISSN 0219-4678, Vol.8, No.4, pp.615-628, 2008

[14] Skala,V.: Geometric Algebra, Extended Cross-product and Laplace Transform for Multidimensional Dynamical Systems, Cybernetics Approaches in Intelligent Systems, Computational Methods in System and Software 2017 (CoMeSySo), Vol.1,pp.62-75, ISSN 2194-5357, Springer, 2018

[15] Skala,V.: Projective Geometry and Duality for Graphics, Games and Visualization - Course SIGGRAPH Asia 2012, Singapore, ISBN 978-1-4503-1757-3, 2012

[16] Skala,V.: Modified Gaussian Elimination without Division Operations, ICNAAM 2013, Rhodos, Greece, AIP Conf.Proceedings, No.1558, pp.1936-1939, AIP Publishing, 2013

[17] Skala,V.: "Extended Cross-product" and Solution of a Linear System of Equations, Computational Scienceand Its Applications-ICCSA 2016, LNCS 9786, Vol.I, pp.18-35, Springer, China, 2016

[18] Skala,V.: Plücker Coordinates and Extended Cross Product for Robust and Fast Intersection Computation, CGI 2016 Proceedings, ACM, pp.57-60, Greece, 2016

[19] Vince,J.: Geometric Algebra for Computer Science, Springer, 2008

[20] Yamaguchi,F.: Computer Aided Geometric Design: A totally Four Dimensional Approach, Springer Verlag, 2002

[21] Paul's Online Math Notes http://tutorial.math.lamar.edu/Classes/DE/SystemsDE.aspx

[22] Solving Systems of ODEs via the Laplace Transform http://www.maplesoft.com/applications/view.aspx?SID=4721&view=html

[23] GeometryAlgebra – http://geometryalgebra.zcu.cz

[24] Laplace Transfrom https://en.wikibooks.org/wiki/Circuit_Theory/Laplace_Transform

[25] Rosenbrock,H.H.: State-Space and Multivariable Theory. Nelson, 1970