Informatics 2019, IEEE proceedings pp.445-449, ISBN 978-1-7281-3178-8, Poprad, Slovakia, 2019

Modified Radial Basis Functions Approximation Respecting Data Local Features

Jakub Vasta
Department of Computer Science and Engineering
Faculty of Applied Sciences, University of West Bohemia
Plzen, Czech Republic
vastja@students.zcu.cz

Vaclav Skala
Department of Computer Science and Engineering
Faculty of Applied Sciences, University of West Bohemia
Plzen, Czech Republic
skala@kiv.zcu.cz

Michal Smolik
Department of Computer Science and Engineering
Faculty of Applied Sciences, University of West Bohemia
Plzen, Czech Republic
smolik@kiv.zcu.cz

Martin Cervenka
Department of Computer Science and Engineering
Faculty of Applied Sciences, University of West Bohemia
Plzen, Czech Republic
cervemar@kiv.zcu.cz

***Abstract*—This paper presents new approaches for Radial basis function (RBF) approximation of 2D height data. The proposed approaches respect local properties of the input data, i.e. stationary points, inflection points, the curvature and other important features of the data. Positions of radial basis functions for RBF approximation are selected according to these features, as the placement of radial basis functions has significant impacts on the final approximation error. The proposed approaches were tested on several data sets. The tests proved significantly better approximation results than the standard RBF approximation with the random distribution of placements of radial basis functions.**

***Index Terms*—Radial basis function, approximation, inflection points, stationary points, Canny edge detector, curvature**

I. Introduction

The approximation is commonly used and well known technique in many computer science disciplines. This technique can be divided into two groups. The first one is the approximation that use the mesh and its connectivity. Some well known approaches that use the triangulation are [1]–[4]. However, all those approaches need the mesh connectivity, i.e. triangulation, which can be time consuming and difficult to compute for higher dimensions. On the opposite site, the second group are approximation techniques that does not require any mesh, i.e. they are called meshless methods. This paper focuses on this kind of approximation.

In the book [5] is provided an introduction for each of the most important and classic meshless methods along with the complete mathematical formulations. In total, it presents 19 meshless methods in detail with full mathematical formulations showing numerical properties such as convergence, consistency and stability. One example of approximation technique is Kriging [6]. It depends on expressing spatial variation of the property in terms of the variogram, and it minimizes the prediction errors which are themselves estimated. Extension and variations of this method are available in [7]–[9]. Another approximation technique is weighted least square method [10], [11] it is simple because it is based on the well-known standard least squares theory. It is attractive because it allows one to directly use the existing body of knowledge of the least squares theory and it is flexible because it can be used to a broad field of applications in the error-invariable models. Very similar approach is the LOWESS method [12] which is used for meshless smoothing and approximation of noisy data.

The Radial basis function (RBF) methods have been widely used for approximation of scattered data, recently. A brief introduction to this method is in [13]. Comparison of different radial basis functions is in [14], [15]. The paper [16] presents an approach for large scattered data interpolation. It uses the space subdivision to reduce the computation time and more importantly to reduce the needed memory for RBF approximation. A modification of this algorithm for $3D$ vector field data approximation is presented in [17]. Very important for the final RBF approximation quality is the distribution of radial basis functions. This problem is described and suggested solution in [18], [19]. Many papers also propose a solution to the selection of the best shape parameters of radial basis functions [20]–[24].

II. Radial Basis Functions

The Radial basis functions (RBF) are commonly used for n-dimensional scattered data approximation and interpolation. This approach is used in many areas, e.g. image reconstruction [25], [26], neural networks [27] and surface reconstruction [28], [29]. The task can be stated as follow. Find analytic function for given pairs of values (x_i, h_i), where x_i is a point position in n-dimensional space and h_i is value in this point. For such data it is not possible to use standard approximation and interpolation techniques because lack of

The authors would like to thank their colleagues at the University of West Bohemia, Plzen, for their discussions and suggestions. The research was supported by the projects: Czech Science Foundation (GACR) No. GA17-05534S and partially by SGS 2019-016.

https://doi.org/10.1109/Informatics47936.2019.9119330

knowledge about data connectivity and ordering. Therefore, the RBF approximation has the following attributes:

- Designed for scattered data approximation/interpolation
- Independent of the data dimension
- Not separable, i.e. it is not valid to approximate/interpolate data "dimension by dimension"
- Invariant with respect to euclidean transformations

Hardy [30] proposed RBF interpolation based on interpolation equation:

$$f(\vec{x}) = \sum_{i=1}^{M} \lambda_i \theta \left(||\vec{x} - \vec{x_i}||\right), \tag{1}$$

where $\vec{x_i}$ is data point and λ_i is point weight. $\theta(r_{ij})$ is radial basis function, where $r_{ij} = ||\vec{x} - \vec{x_i}||$. Radial basis function can differ, but they can be divided into two main groups by theirs range of influence, i.e. global and local functions.

RBF approximation/interpolation leads to the linear equation system $A\vec{x} = \vec{b}$, where approximation differ from interpolation only with form of matrix A. Solvability and stability problems were solved for example in [31] [32]. Wright [32] extend original RBF interpolation with polynomial and added more conditions.

A. Radial Basis Functions approximation

RBF approximation is based on point distance in n-dimensional space and is derived from the same equation (2) as interpolation is.

$$f(\vec{x}) = \sum_{i=1}^{M} \lambda_i \theta \left(||\vec{x} - \epsilon_i||\right), \tag{2}$$

where M is number of radial basis functions, λ_i is weight of radial basis function, θ is radial basis function and ϵ_i is placement of radial basis function.

Given set of value pairs $\{\vec{x_i}, h_i\}_1^N$, where $\vec{x_i}$ is point position in n-dimensional space, h_i is value in this point, N is number of given points. $N \ll M$ therefore we obtain over-determined system of linear equations.

$$h_i = f(\vec{x_i}) = \sum_{i=1}^{M} \lambda_i \theta \left(||\vec{x_j} - \epsilon_j||\right)$$
$$i = \{1, \dots, N\}, M \ll N \tag{3}$$

It can be rewritten in matrix form

$$A\vec{\lambda} = \vec{h} \tag{4}$$

This over-determined system of linear equations can be solved by LSE or QR decomposition.

III. Proposed Approach

Radial basis function placement is important factor for approximation error. In this contribution, property of these good points are proposed with the way to find them. First group are extreme points i.e. local/global minimum or maximum. Next group are points of inflection. These points represents changes in data. Another proposed group of points are stationary points of curvature. These points represents extreme curvature values and in some case they are similar to points of inflection. Last group are edge points as known from image processing, because we can look at data as on image depending on data structure or sampling. Search for important points is amended with Halton sequence [13] sampling with special stress on border and corner sampling for covering whole data set. The last step is reduction of points number with nearest neighbour condition.

The Halton sequence is computed using the following formula:

$$Halton(p)_k = \sum_{i=0}^{\lfloor \log_p k \rfloor} \frac{1}{p^{i+1}} \left(\left\lfloor \frac{k}{p^i} \right\rfloor \mod p \right), \tag{5}$$

where p is a prime number, k is the order of the element of the Halton sequence, i.e. $k \in \{1, \dots, n\}$. For generation of random points with Halton distribution in higher dimension, different values of p are used for each dimension.

A. RBF Approximation with Stationary Points

It is known that stationary points are such points that hold equation

$$\frac{\delta f}{\delta x} = 0 \wedge \frac{\delta f}{\delta y} = 0 \tag{6}$$

This condition is not enough to determine whether the point is global/local extreme or just a saddle point. It is necessary to examine Hessian to determine point property. On the other hand it can be seen that saddle points are as important as points of extreme.

Evaluation of partial derivatives and comparing with zero is not optimal way to find stationary points. Better way is to compare given point with its surrounding i.e. masks for minimum, maximum and saddle point can be created.

B. RBF Approximation with Inflection Points

Points of inflection are such points where surface change from convex to concave or the other way round. For points of inflection in continuous space hold that Gauss curvature is equal to zero

$$\kappa_{gauss} = \frac{\frac{\delta^2 f}{\delta x^2}\frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2}{\left(\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1\right)^2} \tag{7}$$

It can be seen, from equations above, that Gauss curvature is zero only when numerator is equal zero, i.e. Hessian matrix determinant is equal to zero.

https://doi.org/10.1109/Informatics47936.2019.9119330

It is not necessary to compute exact curvature value to find point of inflection. We need to find just points where curvature sign change from negative to positive or vice versa. It can be seen that sign of Gauss curvature only depends on numerator sign because denominator is always positive.

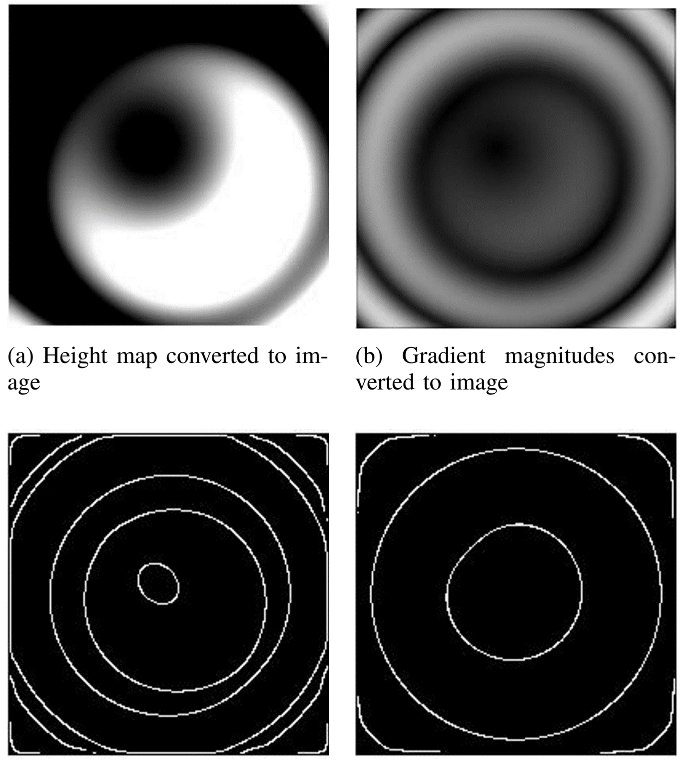
C. RBF Approximation with Stationary Points of Curvature

From (7) we can compute curvature of given surface in every data point. Then it is possible to find stationary points in curvature similar to process described in the section III-A

D. RBF Approximation with Edge Detection

In the simplified scenario, where the points are sampled in the grid pattern, we can look at the data as image i.e. value h on position $[x, y]$ can be considered to be brightness intensity I in pixel (i, j). In case of scatter data there is need to use some special data structure to obtain points adjacency information e.g. kd-tree or adjust used algorithms.

We proposed to detect edges in image i.e. transitions between low and high values. Another suggested approach is to compute data gradient magnitude in each data point and then run edge detector over such field of gradient magnitudes. This approach will find transitions between low and high gradient magnitude areas. To detect edges we can use existing detectors from image processing e.g. Canny, Sobel, Prewitt etc.



(a) Height map converted to image

(b) Gradient magnitudes converted to image

(c) Edge detector over height map

(d) Edge detector over gradient magnitudes map

Fig. 2: The edge detection from height map (a), (c). The edge detection from gradient magnitudes map (b), (d).

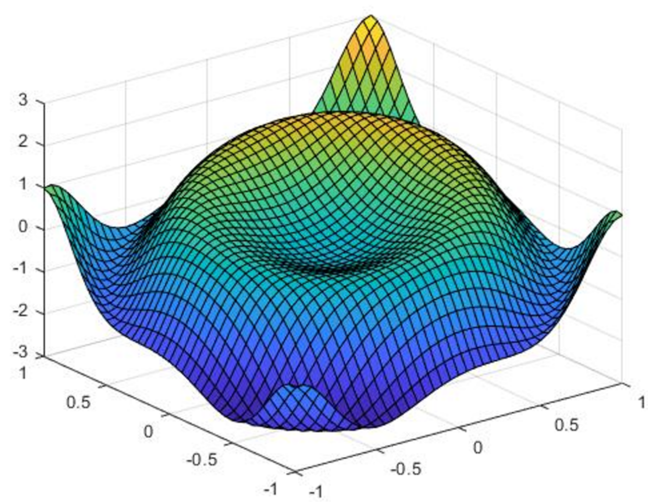


Fig. 1: Test function

$$
\begin{aligned}
f_1(x,y) &= \frac{2}{11}\left(\sin\left(4x^2+4y^2\right) - x + y - \frac{5}{2}\right)\\
f_2(x,y) &= \frac{3}{4}e^{-\frac{1}{4}\left((9x-2)^2+(9y-2)^2\right)}\\
&\quad + \frac{3}{4}e^{-\frac{1}{49}\left((9x+1)^2-\frac{1}{10}(9y+1)^2\right)}\\
&\quad + \frac{1}{2}e^{-\frac{1}{4}\left((9x-7)^2+(9y-3)^2\right)} - \frac{1}{5}e^{-(9x-4)^2-(9y-7)^2}\\
f_3(x,y) &= \frac{1}{9}\tanh\left(9y-9x\right)+1
\end{aligned}
\tag{8}
$$

For comparison was used square mean error per point as can be seen in Fig. 6, Fig. 7 and Fig. 8. In the 1^{st} test function (8) random distribution of placement with Halton sequence provide good results in comparison with other methods, see Fig. 3. This is caused by function shape which fill whole space.

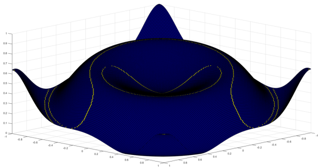
In the 2^{nd} test function (8) can be seen improvement when proposed methods are used because of its special behaviour only in some areas of its domain, see Fig. 4.

The last test function (8) is design to test RBF approximation in general and even with our improvements lot of methods fails, see Fig. 5. What is even more it was found that with proper placement it has no effect on precision to add more points from Halton sequence.

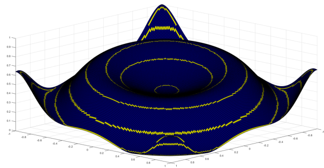
IV. Experimental Results

Proposed methods were tested on several test functions which were designed to represent special data set behaviour. Sampling step is 0.01 and functions are normalized to interval $(x, y, z) \in \langle -1, 1\rangle \times \langle -1, 1\rangle \times \langle 0, 1\rangle$ for comparison purposes. For all functions Gauss radial basis function $\phi(r) = e^{-\epsilon r^2}$ was used.

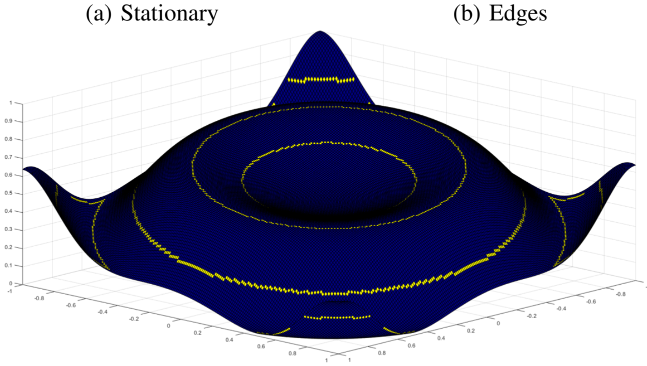
https://doi.org/10.1109/Informatics47936.2019.9119330



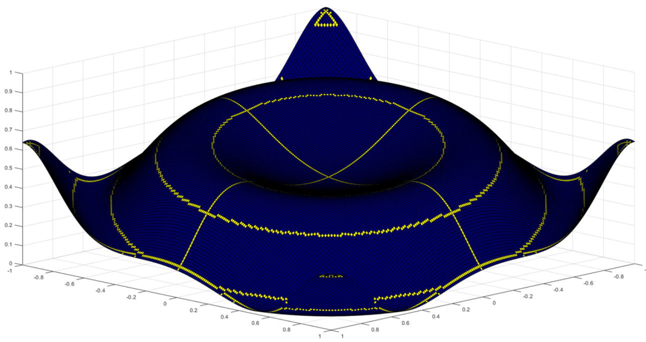
(a) Stationary



(b) Edges

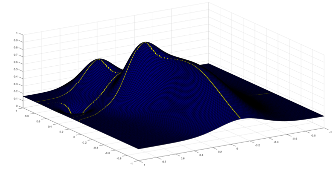


(c) Inflection

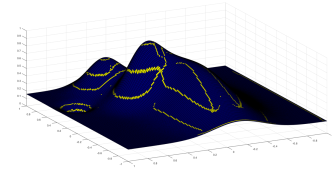


(d) Curvature

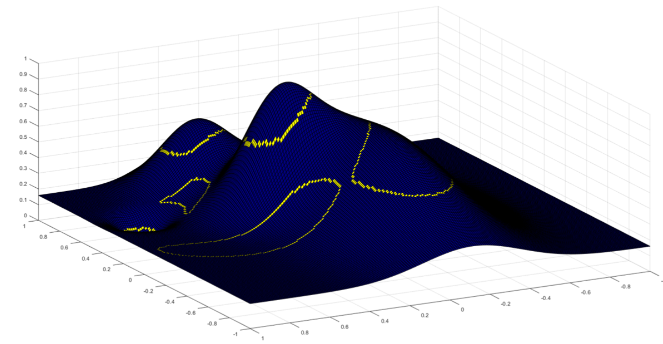
Fig. 3: Methods applied on 1^{st} test function



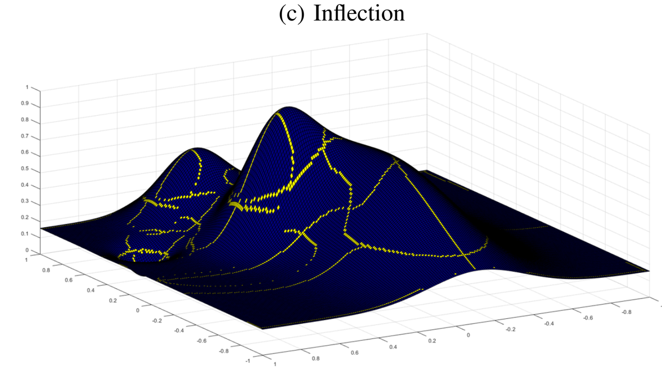
(a) Stationary



(b) Edges



(c) Inflection



(d) Curvature

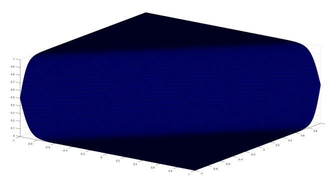
Fig. 4: Methods applied on 2^{nd} test function

V. Conclusion

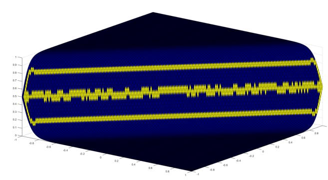
The proposed methods were tested on several standard testing functions, however, only some representative functions are mentioned in this contribution. The above presented methods proved very good results in precision of approximation, even thought some special types of functions, e.g. fast changes, are problematic for all approaches. The experiments also proved validity of the proposed methods for the Radial basis function approximation of scattered data, with regard to a low approximation error with high points reduction leading to a high compression ratio.

Acknowledgment

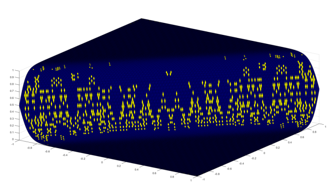
The authors would like to thank their colleagues at the University of West Bohemia, Plzen, especially to colleague Zuzana Majdisova, for their discussions and suggestions. This research was supported by the projects: Czech Science Foundation (GACR) No. GA17-05534S and partially by SGS 2019-016.



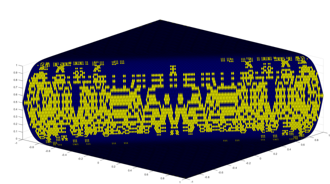
(a) Stationary



(b) Edges



(c) Inflection



(d) Curvature

Fig. 5: Methods applied on 3^{rd} test function

https://doi.org/10.1109/Informatics47936.2019.9119330

Informatics 2019, IEEE proceedings pp.445-449, ISBN 978-1-7281-3178-8, Poprad, Slovakia, 2019

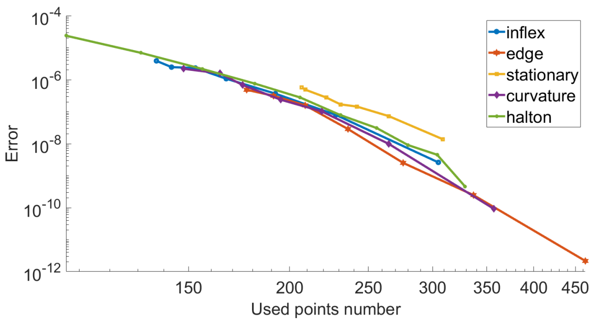


Fig. 6: Mean square error on 1^{st} test function (8)

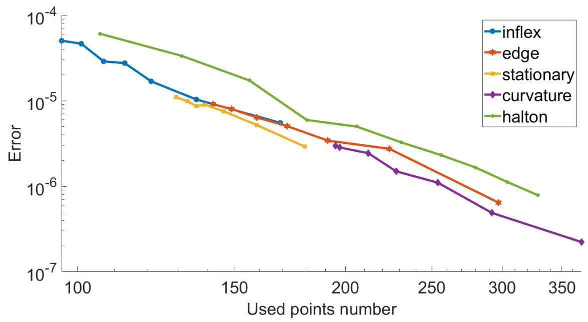


Fig. 7: Mean square error on 2^{nd} test function (8)

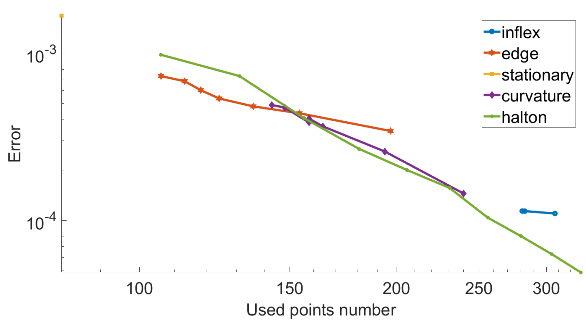


Fig. 8: Mean square error on 3^{rd} test function (8)

REFERENCES

[1] M. Meyer, M. Desbrun, P. Schröder, and A. H. Barr, "Discrete differential-geometry operators for triangulated 2-manifolds," in *Visualization and mathematics III*. Springer, 2003, pp. 35–57.

[2] M. Garland and P. S. Heckbert, "Fast polygonal approximation of terrains and height fields," 1995.

[3] D. Cohen-Steiner, P. Alliez, and M. Desbrun, "Variational shape approximation," in *ACM Transactions on Graphics (ToG)*, vol. 23, no. 3. ACM, 2004, pp. 905–914.

[4] C. L. Bajaj, F. Bernardini, and G. Xu, "Automatic reconstruction of surfaces and scalar fields from 3d scans," in *Computer graphics and interactive techniques*. ACM, 1995, pp. 109–118.

[5] H. Li and S. S. Mulay, *Meshless methods and their numerical properties*. CRC press, 2013.

[6] M. A. Oliver and R. Webster, "Kriging: a method of interpolation for geographical information systems," *International Journal of Geographical Information System*, vol. 4, no. 3, pp. 313–332, 1990.

[7] I. Kaymaz, "Application of kriging method to structural reliability problems," *Structural Safety*, vol. 27, no. 2, pp. 133–151, 2005.

[8] S. Sakata, F. Ashida, and M. Zako, "An efficient algorithm for kriging approximation and optimization with large-scale sampling data," *Computer methods in applied mechanics and engineering*, vol. 193, no. 3-5, pp. 385–404, 2004.

[9] V. R. Joseph, Y. Hung, and A. Sudjianto, "Blind kriging: A new method for developing metamodels," *Journal of mechanical design*, vol. 130, no. 3, p. 031102, 2008.

[10] A. Amiri-Simkooei and S. Jazaeri, "Weighted total least squares formulated by standard least squares theory," *Journal of geodetic science*, vol. 2, no. 2, pp. 113–124, 2012.

[11] T. Asparouhov and B. Muthén, "Weighted least squares estimation with missing data," *Mplus Technical Appendix*, vol. 2010, pp. 1–10, 2010.

[12] M. Smolik, V. Skala, and O. Nedved, "A comparative study of lowess and rbf approximations for visualization," in *ICCSA*. Springer, 2016, pp. 405–419.

[13] G. E. Fasshauer, *Meshfree approximation methods with MATLAB*. World Scientific, 2007, vol. 6.

[14] Z. Majdisova and V. Skala, "Radial basis function approximations: comparison and applications," *Applied Mathematical Modelling*, vol. 51, pp. 728–743, 2017.

[15] ——, "Big geo data surface approximation using radial basis functions: A comparative study," *Computers & Geosciences*, vol. 109, pp. 51–58, 2017.

[16] M. Smolik and V. Skala, "Large scattered data interpolation with radial basis functions and space subdivision," *Integrated Computer-Aided Engineering*, vol. 25, no. 1, pp. 49–62, 2018.

[17] ——, "Efficient simple large scattered 3d vector fields radial basis functions approximation using space subdivision," in *ICCSA*. Springer, 2019, pp. 337–350.

[18] M. Cervenka, M. Smolik, and V. Skala, "A new strategy for scattered data approximation using radial basis functions respecting points of inflection," in *ICCSA*. Springer, 2019, pp. 322–336.

[19] V. Skala and S. Karim, "Finding points of importance for radial basis function approximation of large scattered data," in *Simposium Kebangsaan Sains Metemakik ke 27*, 2019.

[20] J. Wang and G. Liu, "On the optimal shape parameters of radial basis functions used for 2-d meshless methods," *Computer methods in applied mechanics and engineering*, vol. 191, no. 23-24, pp. 2611–2630, 2002.

[21] J. González, I. Rojas, J. Ortega, H. Pomares, F. J. Fernandez, and A. F. Díaz, "Multiobjective evolutionary optimization of the size, shape, and position parameters of radial basis function networks for function approximation," *IEEE Transactions on Neural Networks*, vol. 14, no. 6, pp. 1478–1495, 2003.

[22] S. A. Sarra and D. Sturgill, "A random variable shape parameter strategy for radial basis function approximation methods," *Engineering Analysis with Boundary Elements*, vol. 33, no. 11, pp. 1239–1245, 2009.

[23] Z. Majdisova, V. Skala, and M. Smolik, "Near optimal placement of reference points and choice of an appropriate variable shape parameter for RBF approximation," *Integrated Computer-Aided Engineering*, p. 16, 2019.

[24] V. Skala, S. Karim, and M. Zabran, "Radial basis function approximation optimal shape parameters estimation: Preliminary experimental results," in *Simposium Kebangsaan Sains Metemakik ke 27*, 2019.

[25] K. Uhlir and V. Skala, "Reconstruction of damaged images using radial basis functions," in *European Signal Processing Conference*. IEEE, 2005, pp. 1–4.

[26] J. Zapletal, P. Vaněček, and V. Skala, "Rbf-based image restoration utilising auxiliary points," in *Computer Graphics International Conference*. ACM, 2009, pp. 39–43.

[27] S. Haykin, *Neural networks: a comprehensive foundation*. Prentice Hall PTR, 1994.

[28] R. Pan and V. Skala, "Continuous global optimization in surface reconstruction from an oriented point cloud," *Computer-Aided Design*, vol. 43, no. 8, pp. 896–901, 2011.

[29] ——, "A two-level approach to implicit surface modeling with compactly supported radial basis functions," *Engineering with Computers*, vol. 27, no. 3, pp. 299–307, 2011.

[30] R. L. Hardy, "Multiquadric equations of topography and other irregular surfaces," *Journal of geophysical research*, vol. 76, no. 8, pp. 1905–1915, 1971.

[31] C. A. Micchelli, "Interpolation of scattered data: distance matrices and conditionally positive definite functions," in *Approximation theory and spline functions*. Springer, 1984, pp. 143–145.

[32] G. B. Wright, "Radial basis function interpolation: numerical and analytical developments," 2003.

https://doi.org/10.1109/Informatics47936.2019.9119330