A New Formulation of Plücker Coordinates Using Projective Representation

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***Abstract*—This contribution presents a new formulation of Plücker coordinates using geometric algebra and standard linear algebra with projective representation. The Plücker coordinates are usually used for a line representation in space, which is given by two points. However, the line can be also given as an intersection of two planes in space. The principle of duality leads to a simple formulation for both cases. The presented approach uses homogenous coordinates with the duality principle application. It is convenient for application on GPU as well. The Plücker coordinates are used in many applications, e.g. in robotics, computer aided design and computer graphics algorithms etc.**

Keywords*—*Plücker coordinates, robotics, computer graphics, geometric algebra, projective space, homogeneous coordinates, extended cross-product, duality, GPU.

I. Introduction

The Euclidean coordinates are mostly used in standard formulations of engineering problems. However, the projective extension of the Euclidean space offers higher "flexibility" in problem re-formulations and solutions, especially if the implicit form is used. In addition, the principle of duality, which is not usually considered, can be used to simplify the problem solution or for development of a new solution of a dual problem.

The Plücker coordinates for a line representation in E^3 is quite complex. They are used in many engineering problems, including robotics applications. The Plücker coordinates are formulated in the Euclidean space. The projective extension of the Euclidean space offers actually one parametric representation convenient for implicit formulations of geometric entities. A practical result is, that in many cases it is possible to postpone division operations in numerical computations.

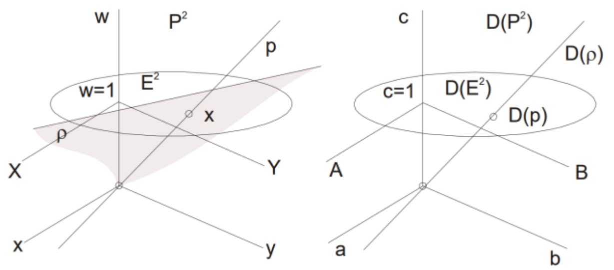

Fig. 1. Projective extension of the Euclidean space and its dual space

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II. Projective Representation

The projective extension of the Euclidean space, i.e. the projective space P^3, is simple, see Fig. 1 for the E^3 case.

The homogeneous coordinates can be used in two different forms in the case of E^3:

$$\boldsymbol{x} = [x_0 : x_1, \ldots, x_3]^T \qquad \boldsymbol{x} = [x, y, z : w]^T \tag{1}$$

The first one is used mostly in mathematically related formulations, the second one in computer graphics related fields. Transformation from the homogeneous coordinates to the Euclidean coordinates is given as

$$X = {}^{x_1}/_{x_0} \qquad Y = {}^{x_2}/_{x_0} \qquad x_0 \neq 0 \tag{2}$$

resp.

$$X = {}^{x}/_{w} \qquad Y = {}^{y}/_{w} \qquad w \neq 0 \tag{3}$$

The first one notation is generally more convenient as the x_0 is actually a "scaling" factor.

It can be seen that a point $\boldsymbol{X} \in E^3$ is actually represented as a line in the projective space $\boldsymbol{x} \in P^3$, without the origin of the projective space.

If geometric entities are described in the implicit form, the projective representation is actually used natively.

Let us consider simple cases, a line in $\boldsymbol{p} \in E^2$ and a plane $\boldsymbol{\rho} \in E^3$, i.e.

$$aX + bY + c = 0 \qquad aX + bY + cZ + d = 0 \tag{4}$$

If the equations are multiplied by $w \neq 0$, then

$$awX + bwY + cw = 0 \tag{5}$$

resp.

$$awX + bwY + cwZ + dw = 0 \tag{6}$$

Using the homogeneous coordinates we get

$$ax + by + cw = 0 \tag{7}$$

resp.

$$ax + bx + cz + dw = 0 \tag{8}$$

Now, the both equations can be rewritten in the vector notation as

$$\boldsymbol{a}^T \boldsymbol{x} = 0 \tag{9}$$

where $\boldsymbol{a} = [a, b : c]^T$ and $\boldsymbol{x} = [x, y : w]^T$,
resp. $\boldsymbol{a} = [a, b, c : d]^T$ and $\boldsymbol{x} = [x, y, z : w]^T$

If $w = 1$ the "standard" form of equations for a line or a plane are obtained.

A distance from the origin of the Euclidean space is given as

$$d = \frac{ax + by + cw}{w\sqrt{a^2 + b^2}} \tag{10}$$

resp.

$$d = \frac{ax + by + cz + dw}{w\sqrt{a^2 + b^2 + c^2}} \tag{11}$$

As a line, resp. a plane is described in linear implicit form, the meaning of symbols $\boldsymbol{a}$ and $\boldsymbol{x}$ can be swapped. This is a consequence of the principle of duality, which will be discussed latter.

Geometric Algebra

Vector algebra (Gibbs algebra) used nowadays actually uses two basic operations on two vectors $\boldsymbol{a}, \boldsymbol{b}$:

- **inner product** (dot or scalar product): $c = \boldsymbol{a} \cdot \boldsymbol{b}$, where the result c is a scalar value
- **outer product**: $\boldsymbol{c} = \boldsymbol{a} \wedge \boldsymbol{b}$, where the result $\boldsymbol{c}$ is a "vector"; $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ in the case of E^3, i.e. cross-product. It should be noted that it is not actually a vector, as it is a *bivector*, which has different properties and it actually represents an oriented area.

In geometrical problems these operations are used very often including their "combinations" like scalar triple product etc.

The Geometric Algebra (GA) is originated in the Clifford algebra using a "new" product type:

- **geometric product**:

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} \tag{12}$$

where $\boldsymbol{ab}$ is the geometric product, $\boldsymbol{a} \cdot \boldsymbol{b}$ is the inner product and $\boldsymbol{a} \wedge \boldsymbol{b}$ is the outer product (in E^3 equivalent to the cross product, i.e. $\boldsymbol{a} \times \boldsymbol{b}$).

If $\boldsymbol{e}_i$ are orthonormal basis vectors in E^3, then

1	0-vector (scalar)
$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$	1-vectors (vectors)
$\boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_3\boldsymbol{e}_1$	2-vectors (bivectors)
$I = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$	3-vector (pseudoscalar)

It can be easily proved that the inner product is

$$\boldsymbol{a} \cdot \boldsymbol{c} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) \tag{13}$$

There is something "strange" in the case of E^3 as the geometric product

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$$

is actually "accumulate" of a scalar value and a result of the outer product, i.e. the cross product E^3, which is a *bivector*, not a vector. The size of it is an area of a rhomboid determined by the $\boldsymbol{a}, \boldsymbol{b}$ vectors in n-dimensional space, in general. Due to the *non-commutativity.*

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a} \qquad \boldsymbol{aa} = \boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2 \tag{14}$$

for all $\boldsymbol{a} \in E^n$.

It means, that there is an inverse defined as

$$\boldsymbol{a}^{-1} = \boldsymbol{a}/|\boldsymbol{a}|^2 \tag{15}$$

There is another "object" called a *blade*. A k-*blade* $\boldsymbol{B}$ is a subspace given by orthogonal vectors $\boldsymbol{e}_{i_1}, \dots, \boldsymbol{e}_{i_k}$, where $\boldsymbol{e}_i \neq \boldsymbol{e}_j$. Similar operations with vectors, operations with k-*blades* are also introduced [4][5][11].

It can be seen that the result contains different elements, i.e. the result $\boldsymbol{c}$ is actually a "multivector".

In the case of GA, vectors are defined as

$$\begin{aligned} \boldsymbol{a} &= (a_1\boldsymbol{e}_1 + \dots + a_n\boldsymbol{e}_n) \\ \boldsymbol{b} &= (b_1\boldsymbol{e}_1 + \dots + b_n\boldsymbol{e}_n) \end{aligned} \tag{16}$$

where $\boldsymbol{e}_i$, $i = 1,..,n$ are basis vectors and $\boldsymbol{c}$, generally a multivector, obtained by multiplying the vectors $\boldsymbol{a}, \boldsymbol{b}$ is actually:

$$\boldsymbol{c} = \boldsymbol{ab} = \sum_{i,j=1}^{n,n} a_i\boldsymbol{e}_i b_j\boldsymbol{e}_j \tag{17}$$

It can be seen that the expression can be split to two different parts

$$\begin{aligned} \sum_{i=1}^{n,n} a_i\boldsymbol{e}_i b_i\boldsymbol{e}_i &= \boldsymbol{a} \cdot \boldsymbol{b} \\ \sum_{\substack{i,j=1 \\ i \neq j}}^{n,n} a_i\boldsymbol{e}_i b_j\boldsymbol{e}_j &= \boldsymbol{a} \wedge \boldsymbol{b} \end{aligned} \tag{18}$$

Note that $\boldsymbol{a} \wedge \boldsymbol{b}$ is actually the cross-product in the case of E^3. It should be noted that geometric algebra is *non-commutative*, i.e.

$$\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i \qquad \boldsymbol{e}_i\boldsymbol{e}_i = 1 \tag{19}$$

Also computation of $\boldsymbol{a} \wedge \boldsymbol{b}$ is a little bit tricky as the GA is non-commutative. It means, that the formula is actually

$$\boldsymbol{a} \wedge \boldsymbol{b} = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i\boldsymbol{e}_i b_j\boldsymbol{e}_j = \sum_{\substack{i,j=1 \\ i > j}}^{n} (a_i b_j - a_j b_i)\boldsymbol{e}_i\boldsymbol{e}_j \tag{20}$$

Let us assume that vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are given in E^3. Then the scalar triple product can be expresses as

$$\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} = q \tag{21}$$

where q is a scalar value, which is called "pseudoscalar" as it has a basis $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$.

III. Geometric Product and Projective Space

In geometry, the outer product (dot or scalar product), i.e. inner product, and cross product, i.e. outer product, are mostly used. However, there is no clear and simple geometric model, what the geometric product actually means, as the result of is a set of objects with different properties and dimensionalities in the E^n case. Also computation of geometric product seems to be more complicated even for the E^3 case and especially if the homogeneous coordinates are to be used.

Geometric product $\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$ of two vectors, using homogeneous coordinates, as $\boldsymbol{a} = [a_1, a_2, a_3 : a_4]^T$ and $\boldsymbol{b} = [b_1, b_2, b_3 : b_4]^T$ can be easily computed using standard matrix operations, respecting anti-commutativity, as:

$$\boldsymbol{ab} \underset{repr}{\Longleftrightarrow} \boldsymbol{ab}^T = \boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{Q} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ b_1a_2 & a_2b_2 & a_2b_3 & a_2b_4 \\ b_1a_3 & b_2a_3 & a_3b_3 & a_3b_4 \\ b_1a_4 & b_2a_4 & b_3a_4 & a_4b_4 \end{bmatrix} = \boldsymbol{B} + \boldsymbol{U} + \boldsymbol{D} \quad (22)$$

where $\boldsymbol{B}, \boldsymbol{U}, \boldsymbol{D}$ are **B**ottom triangular, **U**pper triangular, **D**iagonal matrices, a_4, b_4 are the homogeneous coordinates, i.e. actually w_a, w_b, and $\otimes$ means a tensor anti-commutative product.

TABLE I. GEOMETRIC PRODUCT

$\boldsymbol{ab}$		$\boldsymbol{b}$			
		b_1	b_2	b_3	b_4
$\boldsymbol{a}$	a_1	a_1b_1	a_1b_2	a_1b_3	a_1b_4
	a_2	b_1a_2	a_2b_2	a_2b_3	a_2b_4
	a_3	b_1a_3	b_2a_3	a_3b_3	a_3b_4
	a_4	b_1a_4	b_2a_4	b_3a_4	a_4b_4

Note, that the outer product is anti-commutative as

$$\boldsymbol{e}_i\boldsymbol{e}_j = -\boldsymbol{e}_j\boldsymbol{e}_i \qquad \text{for } i \neq j \quad (23)$$

It can be seen that the diagonal of the matrix $\boldsymbol{Q}$ actually represents the inner product in the projective representation:

$$\boldsymbol{a} \cdot \boldsymbol{b} = [(a_1b_1 + a_2b_2 + a_3b_3): a_4b_4]^T \triangleq \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_4b_4} \quad (24)$$

where $\triangleq$ means projectively equivalent.

The outer product is then represented respecting anti-commutativity as:

$$\boldsymbol{a} \wedge \boldsymbol{b} \underset{repr}{\Longleftrightarrow} \sum_{\boldsymbol{i,j=1}\ \&\boldsymbol{i \neq j}}^{\boldsymbol{3,3}} a_ib_j\boldsymbol{e}_i\boldsymbol{e}_j = \sum_{\boldsymbol{i,j=1}\ \&\ \boldsymbol{i>j}}^{\boldsymbol{3,3}} \left(a_ib_j\boldsymbol{e}_i\boldsymbol{e}_j - b_ia_j\boldsymbol{e}_i\boldsymbol{e}_j\right) = \sum_{\boldsymbol{i,j}\ \&\ \boldsymbol{i>j}}^{\boldsymbol{3,3}} \left(a_ib_j - b_ia_j\right)\boldsymbol{e}_i\boldsymbol{e}_j \quad (25)$$

and it can be recognized a close relation to the Plücker coordinates as well. The outer product can be used for a solution of a linear system of equations [9][10][18], too.

IV. PRINCIPLE OF DUALITY

The principle of duality is an important principle in general. However, its application in geometry in connection with the implicit representation using projective geometry brings some new formulations or even new theorems.. The duality principle for basic geometric entities and operators says [2][3][10][21].

TABLE II. DUALITY OF GEOMETRIC ENTITIES

Duality of geometric entities		
Point in E^2	$\underset{DUAL}{\Longleftrightarrow}$	Line in E^2
Point in E^3	$\underset{DUAL}{\Longleftrightarrow}$	Plane in E^3
Duality of operators		
Union ∪	$\underset{DUAL}{\Longleftrightarrow}$	Intersection ∩

It means, that in E^2 point is dual to a line and vice versa, intersection of two lines is dual to a union of two points, i.e. line given by two points.

As a direct consequence is, that the intersection point $\boldsymbol{x}$ of two lines $\boldsymbol{p}_1, \boldsymbol{p}_2$ is given as

$$\boldsymbol{x} = \boldsymbol{p}_1 \wedge \boldsymbol{p}_2 \quad (26)$$

where $\boldsymbol{x} = [x, y: w]^T$, $\boldsymbol{p}_i = [a_i, b_i: c_i]^T$, $i = 1, 2$.

Due to the principle of duality, as a point is dual to a line etc., a line $\boldsymbol{p}$ passing two given points $\boldsymbol{x}_1, \boldsymbol{x}_2$ is given as

$$\boldsymbol{p} = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \quad (27)$$

where $\boldsymbol{x}_{\mathrm{i}} = [x_i, y_i: w_i]^T$, $i = 1,2$.

If the Euclidean representation is used, the first case lead to a linear system of two equations $\boldsymbol{Ax} = \boldsymbol{b}$, while in the second case to a system $\boldsymbol{Ax} = \boldsymbol{0}$, i.e. homogeneous linear system. This clearly shows one basic advantage of the implicit form with projective representation and the principle of duality use.

In the case of E^3, a point is dual to a plane and vice versa. It means that the intersection point $\boldsymbol{x}$ of three planes $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3$ is given as

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2 \wedge \boldsymbol{\rho}_2 \quad (28)$$

where $\boldsymbol{x} = [x, y, z: w]^T$, $\boldsymbol{\rho}_i = [a_i, b_i, c_i: d_i]^T$, $i = 1, 2,3$.

Due to the principle of duality, as a point is dual to a plane etc., a plane $\boldsymbol{\rho}$ passing three given points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ is given as

$$\boldsymbol{\rho} = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \quad (29)$$

Where $\boldsymbol{x}_{\mathrm{i}} = [x_i, y_i, z_i: w_i]^T$, $i = 1, \ldots, 3$

It can be seen, solutions of those problems are simple and straightforward. If there is a special case or singularity, e.g. two planes are collinear, the homogeneous coordinate of the result $\mathrm{w} = 0$ etc.

However, the question is how to compute an intersection of two planes $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$, which is dual to a line given by two points $\boldsymbol{x}_1, \boldsymbol{x}_2$ in homogeneous coordinates as those problems are dual. More specifically, a line $\boldsymbol{p}$ given as an intersection of two planes $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$ is given as

$$\boldsymbol{p} = \boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2 \quad (30)$$

resp.

$$\boldsymbol{p} = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \quad (31)$$

Note, that the line $\boldsymbol{p} \in E^3$

The solution can be easily obtained using Plücker coordinates [19][20][21].

V. Plücker Coordinates

The Plücker coordinates are an alternative to a parametric form of a line in E^3

$$\boldsymbol{p}(t) = \boldsymbol{x}_1 + \boldsymbol{s}\, t \qquad t \in (-\infty, \infty) \tag{32}$$

The advantage of the Plücker coordinates is that the reference point is the closest point to the origin of the coordinate system natively and offers a momentum as well.

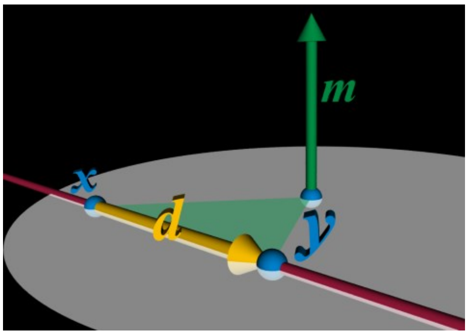

Fig. 1. The Plücker coordinates:Displacement and moment of two points on line (taken from [23])

The Plücker coordinates of a line given by two points $\boldsymbol{x}_1 = [x_1, y_1, z_1 : w_1]^T$ and $\boldsymbol{x}_2 = [x_2, y_2, z_2 : w_2]^T$ are determined as [7][21];

$$\begin{aligned} l_{41} &= w_1 x_2 - w_2 x_1 \qquad & l_{23} &= y_1 z_2 - y_2 z_1 \\ l_{42} &= w_1 y_2 - w_2 y_1 \qquad & l_{31} &= z_1 x_2 - z_2 x_1 \\ l_{43} &= w_1 z_2 - w_2 z_1 \qquad & l_{12} &= x_1 y_2 - x_2 y_1 \end{aligned} \tag{33}$$

or in a matrix form

$$\boldsymbol{L} = \boldsymbol{x}_1 \boldsymbol{x}_2^T - \boldsymbol{x}_2 \boldsymbol{x}_1^T \tag{34}$$

Let us define two vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}$ as

$$\boldsymbol{\omega} = [l_{41}, l_{42}, l_{43}]^T \qquad \boldsymbol{v} = [l_{23}, l_{31}, l_{12}]^T \tag{35}$$

It means that $\boldsymbol{\omega}$ represents the "directional" vector, while $\boldsymbol{v}$ represents the "positional" vector.

It can be seen, that the matrix $\boldsymbol{L}$ is closely related to computation of $\boldsymbol{a} \wedge \boldsymbol{b}$.

In the case of the Euclidean space ($w = 1$), we get:

$$\boldsymbol{\omega} = \boldsymbol{X}_2 - \boldsymbol{X}_1 \qquad \boldsymbol{v} = \boldsymbol{X}_2 \times \boldsymbol{X}_1 \tag{36}$$

where $\boldsymbol{X}_i = [X_i, Y_i, Z_i]^T$ are points in the Euclidean space.

In general case, when $w \neq 1 \;\&\; w \neq 0$, the line using the Plucker coordinates is defined as [21][22]:

$$\boldsymbol{q}(t) = \frac{\boldsymbol{\omega} \times \boldsymbol{v}}{\|\boldsymbol{\omega}\|^2} + \boldsymbol{\omega} t \qquad t \in (-\infty, \infty) \tag{37}$$

Due to the principle of duality, the same formula can be applied for the intersection of two planes $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$. Therefore:

$$\boldsymbol{q}(t) = \frac{\boldsymbol{\omega} \times \boldsymbol{v}}{\|\boldsymbol{\omega}\|^2} + \boldsymbol{\omega} t \tag{38}$$

using

$$\boldsymbol{L} = \boldsymbol{\rho}_1 \boldsymbol{\rho}_2^T - \boldsymbol{\rho}_2 \boldsymbol{\rho}_1^T \tag{39}$$

where $\boldsymbol{\rho}_i = [a_i, b_i, c_i : d_i]^T, i = 1, 2$.

However, it should be noted that:

- If projective representation for the line $\boldsymbol{q}(t)$ is used, then division operation can be eliminated and the line can be represented as

$$\widetilde{\boldsymbol{q}(t)} = [\boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\omega}\|\boldsymbol{\omega}\|^2 t : \|\boldsymbol{\omega}\|^2]^T \tag{40}$$

- Computation of the line using the $\boldsymbol{L}$ matrix is not efficient. Using GA formulation approx. 50% of computation can be saved, as computation of $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2$, resp. $\boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2$ is more efficient and *vector-vector* operations can be used.

As the vectors have 4 elements, there is a native use of GPU or SSE instructions use in the case of GA approach, which gives additional speed and simplicity of formulations.

Let us consider two points $\boldsymbol{x}_1 = [x_1, y_1, z_1 : w_1]^T$ and $\boldsymbol{x}_2 = [x_2, y_2, z_2 : w_2]^T$ again and explore their geometric product, i.e.

$$\boldsymbol{x}_1 \boldsymbol{x}_2 \triangleq \boldsymbol{x}_1 \boldsymbol{x}_2^T = \boldsymbol{x}_1 \otimes \boldsymbol{x}_2 = \boldsymbol{Q} = \begin{bmatrix} x_1 x_2 & x_1 y_2 & x_1 z_2 & x_1 w_2 \\ y_1 x_2 & y_1 y_2 & y_1 z_2 & y_1 w_2 \\ z_1 x_2 & z_1 y_2 & z_1 z_2 & z_1 w_2 \\ w_1 x_2 & w_1 y_2 & w_1 z_2 & w_1 w_2 \end{bmatrix} = \boldsymbol{B} + \boldsymbol{U} + \boldsymbol{D} \tag{41}$$

A line $\boldsymbol{p}$ passing those two points is given as

$$\boldsymbol{p} = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \tag{42}$$

Generally, the geometric product is *anti-commutative*, some additional care is to be taken. It can be also represented by a table:

TABLE III. Geometric Product in Projective Space - Points

$\boldsymbol{x}_1 \boldsymbol{x}_2$		$\boldsymbol{x}_2$			
		x_2	y_2	z_2	w_2
$\boldsymbol{x}_1$	x_1	$x_1 x_2$	$x_1 y_2$	$x_1 z_2$	$x_1 w_2$
	y_1	$y_1 x_2$	$y_1 y_2$	$y_1 z_2$	$y_1 w_2$
	z_1	$z_1 x_2$	$z_1 y_2$	$z_1 z_2$	$z_1 w_2$
	w_1	$w_1 x_2$	$w_1 y_2$	$w_1 z_2$	$w_1 w_2$

Now, comparing the Plücker coordinates l_{ij} we can see that the non-diagonal elements of the matrix $\boldsymbol{Q}$, i.e. matrices $\boldsymbol{B}, \boldsymbol{U}$ represents all parts of the Plücker coordinates.

It should be noted, that the meaning of the diagonal matrix $\boldsymbol{D}$ is actually a projective representation of the inner product (dot product) of those two vectors $\boldsymbol{x}_1, \boldsymbol{x}_2$.

It can be seen that the matrix $\boldsymbol{Q}$ represents geometric product of two vectors in the projective space.

VI. Duality and Plücker Coordinates

The principle of duality explained earlier can be applied as well. As a point is dual to a plane in E^3, we can easily solve a problem of intersection of two planes.

Let us imagine three planes $\boldsymbol{\rho}_i = [a_i, b_i, c_i : d_i]^T$ $i = 1, \dots, 3$. The point of their intersection is given as

$$\boldsymbol{x} = \boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2 \wedge \boldsymbol{\rho}_3 \tag{43}$$

where $\boldsymbol{x} = [x, y, z : w]^T$ is a point of intersection. The singular cases lead to $w = 0$ and type of singularity is to be determined consequently.

However, intersection computation of two planes $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$ is a little bit more complicated, if the Euclidean space is used. Using the principle of duality we can write a solution directly using the Plücker coordinates. However, using the GA , we can directly write

$$\boldsymbol{p} = \boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2 \qquad (44)$$

TABLE IV. GEOMETRIC PRODUCT IN PROJECTIVE SPACE - PLANES

$\boldsymbol{\rho}_1\boldsymbol{\rho}_2$		$\boldsymbol{\rho}_2$			
		a_2	b_2	c_2	d_2
$\boldsymbol{\rho}_1$	a_1	a_1a_2	a_1b_2	a_1c_2	a_1d_2
	b_1	b_1a_2	b_1b_2	b_1c_2	b_1d_2
	c_1	c_1a_2	c_1b_2	c_1c_2	b_1d_2
	d_1	d_1a_2	d_1b_2	d_1c_2	d_1d_2

The table above is actually a result of the geometric product of two planes $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$

$$\begin{aligned}\boldsymbol{\rho}_1\boldsymbol{\rho}_2 &\triangleq \boldsymbol{\rho}_1\boldsymbol{\rho}_2^T = \boldsymbol{\rho}_1 \otimes \boldsymbol{\rho}_2 = \boldsymbol{Q} \\ &= \begin{bmatrix} a_1a_2 & a_1b_2 & a_1c_2 & a_1d_2 \\ b_1a_2 & b_1b_2 & b_1c_2 & y_1d_2 \\ c_1a_2 & c_1b_2 & c_1c_2 & z_1d_2 \\ d_1a_2 & d_1b_2 & d_1c_2 & d_1d_2 \end{bmatrix} \\ &= \boldsymbol{B} + \boldsymbol{U} + \boldsymbol{D}\end{aligned} \qquad (45)$$

It means that we have computation of the Plücker coordinates for the both cases, i.e. for computation of a line $\boldsymbol{p} = \boldsymbol{\rho}_1 \wedge \boldsymbol{\rho}_2$ or $\boldsymbol{p} = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2$ is given as:

- a union of two points in E^3 and
- an intersection of two planes in E^3

using the projective representation and the principle of duality.

It should be noted that the given approach offers:

- significant simplification of computation of the Plücker coordinates as it is simple and easy to derive and explain,
- uses *vector-vector* operations, which is especially convenient for SSE and GPU application
- one code sequence for the both cases

As the Plücker coordinates are also in mechanical engineering applications, especially in robotics due to its simple displacement and momentum specifications, and in other fields simple explanation and derivation is another very important argument for GA approach application.

VII. CONCLUSION

In this contribution, a new reformulation of the Plücker coordinates is presented based on geometric product. It uses geometric algebra approach and standard linear algebra with projective representation. Application of the principle of duality leads to a simple formulations for the both cases, i.e. for the line given by two points or by two intersection points in E^3 . The proposed approach is convenient for GPU application as well, as formulation is based on *vector-vector* operations.

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