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Novel RBF Approximation Method Based on Geometrical Properties for Signal Processing with a New RBF Function: Experimental Comparison

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***Abstract*—Interpolation and approximation methods are widely used in many areas. They can be divided to methods based on meshing (tessellation) of the data domain and to meshless (meshfree) methods, which do not require the domain tessellation of scattered data. Scattered n-dimensional data radial basis function (RBF) interpolation and approximation leads to a solution of linear system of equations.**

This contribution presents a new approach to the RBF approximation based on analysis of geometrical properties of signals, i.e. sampled curves. Also a newly developed radial basis function was used and proved better precision of approximation.

Experimental comparison of several RBF functions (Gauss, Thin-Plate Spline, CS-RBF and a new proposed RBF) is described with analysis of their properties. Special attention was taken to the precision of approximation and conditionality issues. The proposed approach can be extended to a higher dimensional case and for vector data, e.q. fluid flow, too.

***Index Terms*—Approximation, Radial basis functions, RBF, Signal processing, Computer graphics, Meshless methods.**

I. Introduction

Interpolation and approximation of scattered data is required in many areas. As there is no ordering defined for d-dimensional case, if $d \geq 1$, usually two approaches are taken:

- Tessellation of the data domain, e.g. using Delaunay triangulation and application of a selected interpolation or approximation method. However, the Delaunay tessellation has a computational complexity $O\left(n^{\lceil d^2/2 \rceil}\right)$. This leads high computational complexity and to implementation problems in the case that $d > 2$. Another computational problems can be expected, if smoothness of the interpolation or approximation is required.
- Use of meshless methods based on Radial Basis Functions (RBF) use leads to a solution of a linear system of equations $\mathbf{Ax} = \mathbf{b}$, in general, and the computational complexity is nearly independent from the dimensionality of the data domain, see Hardy [1], . Even more, if relevant RBF function is selected, higher degree of smoothness is obtained. On the other hand, interpolation and approximation methods based on meshless approach usually have a problem with a precision on borders or on discontinuities, in general. The meshless methods can be also used for approximation of vector data, i.e. fluid flow etc. However, some RBFs applications might lead to numerical problems due to ill-conditioned matrix of the linear system, especially for large data sets.

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Usually, the approximation methods use a general method, which is not taking directly geometrical properties of the signal into account, e.g. Skala & Smolik & Nedved [2]. However, if some information on signal geometry can be extracted from data and used, the approximation should be more precise and simpler. Such approach has been used by Majdisova & Skala & Smolik [3]. This approach seems to be quite complicated, as it is based on properties of cubic curve in floating data window.

This contribution is focused on the following main aspects:

- how geometrical properties of a signal can be efficiently used for good and robust approximation,
- how to approximate a signal, i.e. a sampled curves, with a good precision with a minimal number of radial basis functions (RBFs),
- what kind of RBFs probably gives better results,
- what are properties of a newly developed RBF in terms of precision and numerical precision.

As some RBF functions have a parameter, called a shape parameter, some problems can be expected with an optimal shape parameter selection or estimation. Some proposal how to select suitable shape parameters were introduced by Karageorghis [4], Wang & Liu [5] for range of shape parameters generation, Afiatdoust & Esmaeilbeigi [6] presented use of genetic algorithm. Moreover, Sarra & Sturgill [7] propose non-deterministic approach based on random shape parameter generation. However, some experiments recently made by Skala & Karim & Zabran [8] proved, that there is probably no optimal constant shape parameter nor one vector of optimal shape parameters for each RBF.

Another problem is shape parameter selection Karageorghis [4]. There are two particular cases which may occur in general: approximation will be inaccurate or the problem may

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become ill-conditioned. That is why correct shape parameter selection is needed. Some approaches to select suitable shape parameters already exists. Approach introduced by Wang & Liu [5] for example generates range of shape parameters, Afiatdoust & Esmaeilbeigi [6] presented theirs approach using genetic algorithm. Moreover, Sarra & Sturgill [7] propose non-deterministic approach based on random generator. Some recent research have been devoted to variable shape parameters, i.e, each RBF function has a different shape parameter, e.g. Majdisova [3], Skala [8].

Application of the RBF interpolation and approximation in engineering practice can be found in Biancolini [9], Fasshauer [10], Menandro [11] . Also RBFs are used for vector field interpolation and approximation, e.g. Smolik [12], [13], [14], [15], Skala [16], solution of partial differential equations (PDE) Zhang citezhang and Neural Networks RBF Zhang [17], Yinwey [18]. Comparison of selected RBFs can be found in Majdisova [19]

II. RBF INTERPOLATION

According to Hardy [20], RBF interpolation is based on determining the distance of two point (in the d-dimensional space in general). The interpolation is given in the form:

$$h\left(\mathbf{x}\right)=\sum_{j=1}^{N}\lambda_j\varphi\left(\left|\left|\mathbf{x}-\mathbf{x}_j\right|\right|\right)=\sum_{j=1}^{N}\lambda_j\varphi\left(r_i\right) \tag{1}$$

where r_i is the distance from a point x to the point x_i. As the parameter of the function φ is a distance of two points in the d-dimensional space, the interpolation is non-separable by a dimension. The RFBs will be described in detailed latter on.

For each point x_i the interpolating function has to have value h_i. Therefore, we are getting a system of linear equations:

$$h\left(\mathbf{x}_i\right)=\sum_{j=1}^{N}\lambda_j\varphi\left(\left|\left|\mathbf{x}_i-\mathbf{x}_j\right|\right|\right)=\sum_{j=1}^{N}\lambda_j\varphi\left(r_{ij}\right) \tag{2}$$

where λ_j are unknown weights for each radial basis function, N is the number of given points and φ is the radial basis function itself. It can be rewritten if the matrix form as:

$$\mathbf{A}\lambda=\mathbf{h} \tag{3}$$

or in a detailed form as (4):

$$\begin{bmatrix}\varphi_{11} & \cdots & \varphi_{1j} & \cdots & \varphi_{1N}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \varphi_{i1} & \cdots & \varphi_{ij} & \cdots & \varphi_{iN}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ \varphi_{N1} & \cdots & \varphi_{Nj} & \cdots & \varphi_{NN}\end{bmatrix}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_i\\ \vdots\\ \lambda_N\end{bmatrix}=\begin{bmatrix}h_1\\ \vdots\\ h_i\\ \vdots\\ h_N\end{bmatrix} \tag{4}$$

After solving the system of linear equations, interpolated value at the point $\mathbf{x}$ is computed using (1). However, due to numerical robustness and stability, additional polynomial conditions are usually added Skala [21] [22]. In the case of an additional polynomial we obtain:

$$h\left(\mathbf{x}_i\right)=\sum_{j=1}^{N}\lambda_j\varphi\left(\left|\left|\mathbf{x}_i-\mathbf{x}_j\right|\right|\right)+P_k(\mathbf{x}_i) \tag{5}$$

In the case of a linear polynomial:

$$P_k\left(\mathbf{x}\right)=a_0+a_1x+a_2y \tag{6}$$

This additional conditions can be rewritten as:

$$\sum_{j=1}^{N}\lambda_j=0 \qquad \sum_{j=1}^{N}\lambda_j\mathbf{x}_j=0 \tag{7}$$

$$\begin{bmatrix}\varphi_{11} & \dots & \varphi_{1N} & 1 & x_1 & y_1\\ \vdots & \ddots & \vdots & 1 & \vdots & \vdots\\ \varphi_{N1} & \dots & \varphi_{NN} & 1 & x_N & y_N\\ 1 & 1 & 1 & 1 & 0 & 0\\ x_1 & \dots & x_N & 1 & 0 & 0\\ y_1 & \dots & y_N & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_1\\ a_0\\ a_1\\ a_2\end{bmatrix}=\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_1\\ 0\\ 0\\ 0\end{bmatrix} \tag{8}$$

This matrix can be further rewritten in more compact way:

$$\begin{bmatrix}\mathbf{A} & \mathbf{P}\\ \mathbf{P^T} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\lambda\\ \mathbf{a}\end{bmatrix}=\begin{bmatrix}\mathbf{h}\\ \mathbf{0}\end{bmatrix} \tag{9}$$

The matrix $\mathbf{P}$ represents polynomial polynomial additional conditions, λ is vector of RBF weights, vector $\mathbf{a}$ contains resulting polynomial coefficients and $\mathbf{h}$ are given values at the given points.

It should be noted that in some cases that it can be counterproductive especially for large scope of domain data Skala [21] [22].

III. RBF APPROXIMATION

Approximation methods are slightly different from interpolation as the final approximated curve does not need to "pass" all the given points. If the matrix $\mathbf{A}$ is a square matrix (RBF count M is equal to size of the $\mathbf{x}$ vector), this is an interpolation problem. On the other hand, when M is smaller, it becomes approximation problem, because equation system became over-determined. Let us ξ_j are significant points in the given signal, then the approximation can be formulated as:

$$h\left(\mathbf{x}_i\right)=\sum_{j=1}^{M}\lambda_j\varphi\left(\left|\left|\mathbf{x}_i-\xi_j\right|\right|\right) \tag{10}$$

where λ_j are weights of each radial basis function, M is count of RBF being used and $M \ll N$, the φ is the RBF, ξ_j are center (important) points. Then the approximation can be expressed by equation (11):

$$\begin{bmatrix}\varphi_{11} & \cdots & \varphi_{1M}\\ \vdots & \ddots & \vdots\\ \varphi_{i1} & \cdots & \varphi_{iM}\\ \vdots & \ddots & \vdots\\ \varphi_{N1} & \cdots & \varphi_{NM}\end{bmatrix}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_M\end{bmatrix}=\begin{bmatrix}h_1\\ \vdots\\ h_i\\ \vdots\\ h_N\end{bmatrix} \tag{11}$$

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Overdetermined linear equation system is to be solved, e.q. by the Least Square Error method (12), which minimizes the mean square error.

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{h} \tag{12}$$

However, in the approximation case, the additional polynomial conditions cannot be used Majdisova [23], [24].

IV. RBF FUNCTIONS

There are several radial basis functions Fasshauer [10], Majdisova [25] [26]. They can be divided into two major groups (Smolik [12]):

- ”global” RBFs having global influence, e.g. $r^2lg(r)$, $exp(-\alpha r^2)$, $sqrt(\alpha + r^2)$ $1/sqrt(\alpha + r^2)$, $1/(\alpha + r^2)$, etc. where α is a shape parameter. The RBF matrix is usually full and ill conditioned.
- ”local” RBFs - Compactly Supported RBF (CS-RBF) have a non-zero value for the interval $\langle 0, 1\rangle$ only. The RBF matrix is usually sparse as it depends on the scaling of the interval $\langle 0, 1\rangle$ to the required one. Some examples are listed in Tab.I and they are shown on Fig.1.

Some examples of RBFs are listed in Tab.I and at Fig.1.

ID	Function
1	$(1-\hat{r})_+$
2	$(1-\hat{r})_+^3\,(3r+1)$
3	$(1-\hat{r})_+^5\,(8r^2+5r+1)$
4	$(1-\hat{r})_+^2$
5	$(1-\hat{r})_+^4\,(4r+1)$
6	$(1-\hat{r})_+^6\,(35r^2+18r+3)$
7	$(1-\hat{r})_+^8\,(32r^3+25r^2+8r+3)$
8	$(1-\hat{r})_+^3$
9	$(1-\hat{r})_+^3\,(5r+1)$
10	$(1-\hat{r})_+^7\,(16r^2+7r+1)$

TABLE I: List of well-known CS-RBF.

V. PROPOSED RBF

In this paper a new global radial basis function is proposed. It has one shape parameter and is defined as is in (13).

$$\varphi(r) = r^2\left(r^\alpha - 1\right) \tag{13}$$

where α is a shape parameter. The function is shown at Fig.2.

VI. DESCRIPTION OF EXPERIMENTS

For sake of simplicity, all signal values has been normalized to interval $h(\mathbf{x}_i) \in \langle 0, 1\rangle$. Signal domain has been set to the $\mathbf{x}_i \in \langle 0, 1\rangle$ as well for the same reason.

As already mentioned, four RBF has been used for testing purposes. Gaussian RBF (in equation (14)) is global radial basis function with one shape parameter α defining its dispersion.

$$\varphi(r) = e^{-\alpha r^2} \tag{14}$$

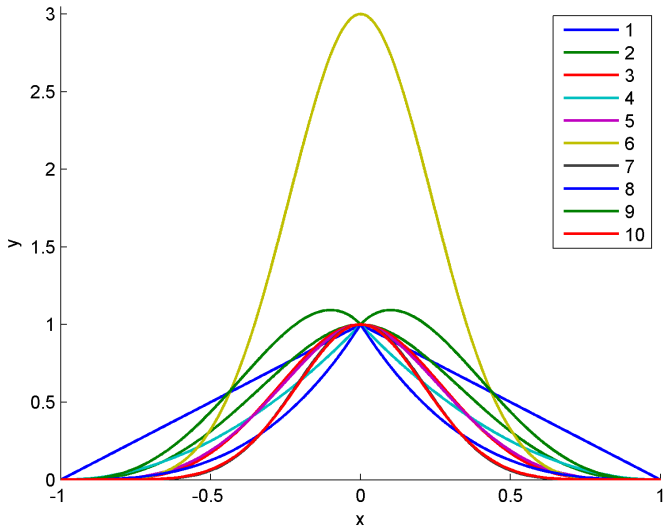

Fig. 1: Plotted CSRBFs taken from [12].

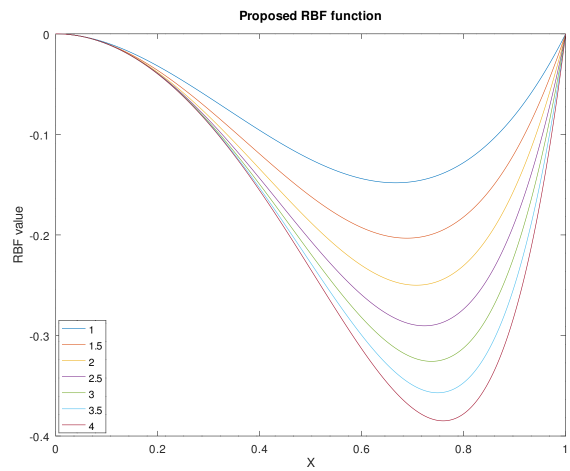

Fig. 2: Proposed RBF function with various shape parameters.

Thin plate spline (TPS) function is the next one which has been tested. It is global RBF as well, and it is defined in (15).

$$\varphi(r) = r^2 \log r \tag{15}$$

Next class of function on the list is CS-RBF. In particular, (16) function has been selected from the Tab.I. It is worth noting that this function (like all CS-RBF) is local.

$$\varphi(r) = (1-\hat{r})_+^7\left(16r^2 + 7r + 1\right) \tag{16}$$

Last but not least we propose another RBF. It is global RBF with one shape parameter α and it is described by equation (13).

Described radial basis functions approximation has been tested against multiple signals, however, there are listed only some of tested signals in this paper. This signal subset contains following functions:

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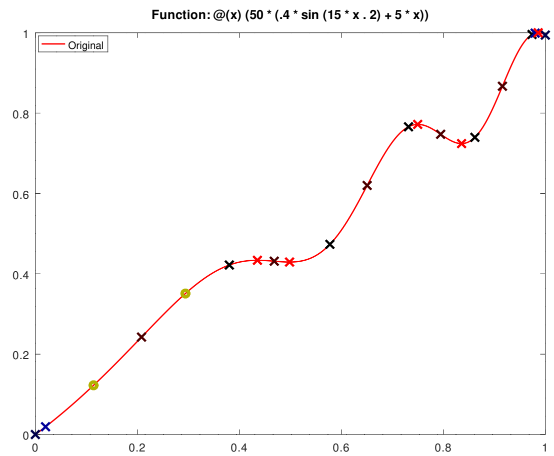

Fig. 3: Function 1.

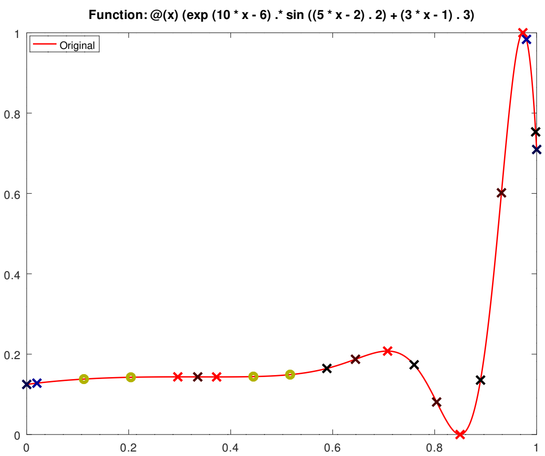

Fig. 5: Function 3.

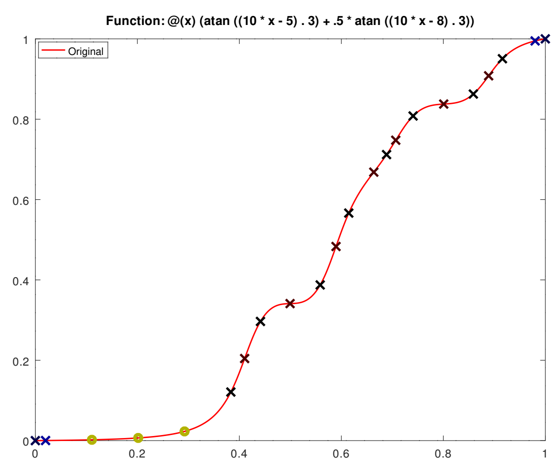

Fig. 4: Function 2.

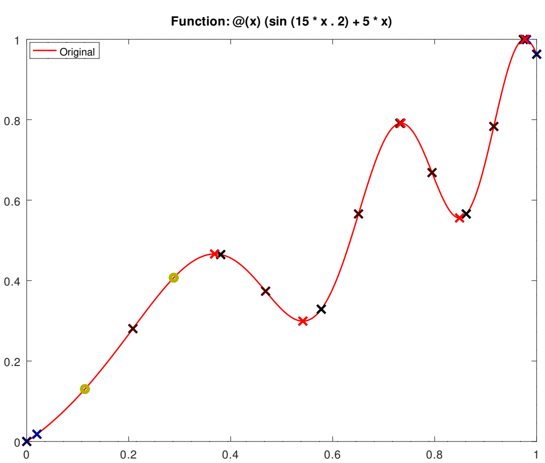

Fig. 6: Function 4.

1) $50\left(0.4sin\left(15x^2\right)+5x\right)$
2) $atan\left(\left(10x-5\right)^3\right)+0.5atan\left(\left(10x-8\right)^3\right)$
3) $e^{10x-6}sin\left(\left(5x-2\right)^2\right)+\left(3x-1\right)^3$
4) $sin\left(15x^2\right)+5x$

Selected signals (sampled curves) are shown at Fig.3, Fig.4, Fig.5 and Fig.6.

The RBF center points ξ_i are shown as various marks on plotted signal curve on each plot.

Four already mentioned RBFs (Gauss, TPS, CS-RBF and the proposed one) were selected to approximate these signals (among others).

VII. Experimental results

The proposed approach was tested on several testing functions, see Tab.II. It should be noted explicitly, that all function were normalized for the interval $x \in \langle 0,1\rangle, y \in \langle 0,1\rangle$. In order to easily compare errors of the proposed RBF approximation methods.

As the proposed RBF approximation is based on finding significant geometric properties, such as maxima, minima, inflection points, etc., the conditionality of the RBF metrics and mutual comparison of errors were analyzed. Especially error behaviour is considered as the critical issue in approximation in general.

In this contribution only couple of function used are presented, see Fig.3, Fig.4, Fig.5 and Fig.6. The relevant approximation error behaviour is presented at Fig.7, Fig.8, Fig.9 and Fig.10.

Exact numerical experimental results are presented in following Tab.III (mean square error), Tab.IV (maximum absolute error) and Tab.V (equation system matrix condition numbers) respectively. It can be seen that the high error is caused by significant under-sampling. Inclusion of additional point(s) leads to significant decrease of the approximation error.

It should be noted, that this contribution is analyzing the approximation behaviour at the lowest border of the sampling frequency.

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1	$\sin\left(15x^2\right)+5x$
2	$0.5\cos\left(20x\right)+5x$
3	$50\left(0.4\sin\left(15x^2\right)+5x\right)$
4	$\sin\left(8\pi x\right)$
5	$\sin\left(6\pi x^2\right)$
6	$\sin\left(25x+0.1\right)/\left(25x+0.1\right)$
7	$2\sin\left(2\pi x\right)+\sin\left(4\pi x\right)$
8	$2\sin\left(2\pi x\right)+\sin\left(4\pi x\right)+\sin\left(8\pi x\right)$
9	$-2\sin\left(2\pi x\right)+\cos\left(6\pi x\right)$
10	$2\sin\left(2\pi x\right)+\cos\left(6\pi x\right)$
11	$-2\sin\left(2\pi x\right)+\cos\left(6\pi x\right)-x$
12	$-2\cos\left(2\pi x\right)-\cos\left(4\pi x\right)$
13	$atan\left(\left(10x-5\right)^3\right)+0.5atan\left(\left(10x-8\right)^3\right)$
14	$\left(4.48x-1.88\right)\sin\left(\left(4.88x-1.88\right)^2\right)+1$
15	$e^{10x-6)}\sin\left(\left(5x-2\right)^2\right)+\left(3x-1\right)^3$
16	$\left(1/9\right)tanh\left(9x+0.5\right)$

TABLE II: Tested artificial signals.

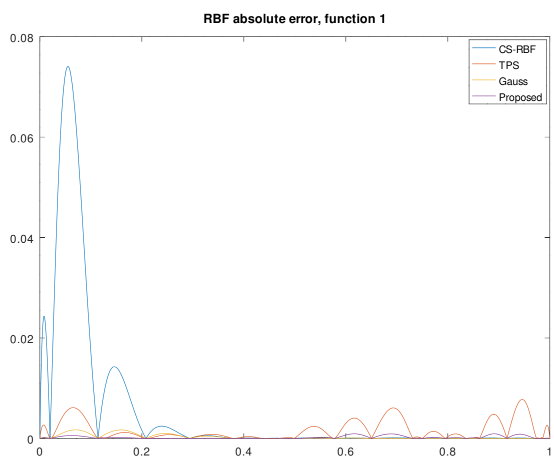

Fig. 7: Function 1 differences.

Tab.III, Tab.IV and Tab.V present the error behaviour numerically.

Function number	Radial basis function			
	CS-RBF	TPS	Gauss	Proposed
1	$2.41{\cdot}10^{-4}$	$6.40{\cdot}10^{-6}$	$3.13\cdot10^{-7}$	$1.21\cdot10^{-7}$
2	$1.54{\cdot}10^{-1}$	$3.95{\cdot}10^{-6}$	$2.56\cdot10^{-2}$	$2.34\cdot10^{-6}$
3	$9.21{\cdot}10^{-4}$	$8.67{\cdot}10^{-6}$	$2.51\cdot10^{-4}$	$3.30\cdot10^{-7}$
4	$7.92{\cdot}10^{-4}$	$2.56{\cdot}10^{-5}$	$3.12\cdot10^{-7}$	$5.23\cdot10^{-7}$

TABLE III: Mean square error.

The experiments proved that the sampled curves can be efficiently approximated by the few "important" points, i.e. extrema, inflections etc., with acceptable low error. The proposed method also leads to good data compression.

Experiments also proved that application of the CS-RBFs with a constant shape parameter is not convenient, unless addi-

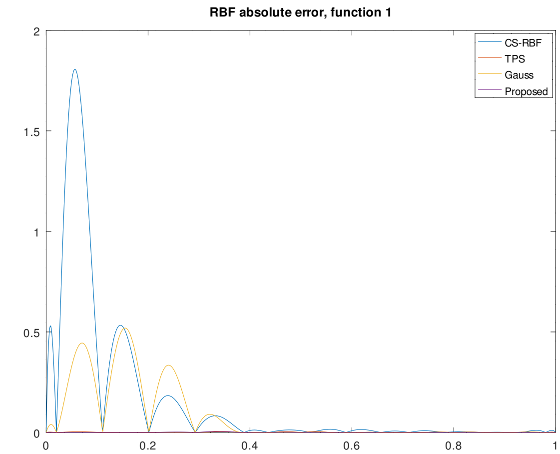

Fig. 8: Function 2 differences.

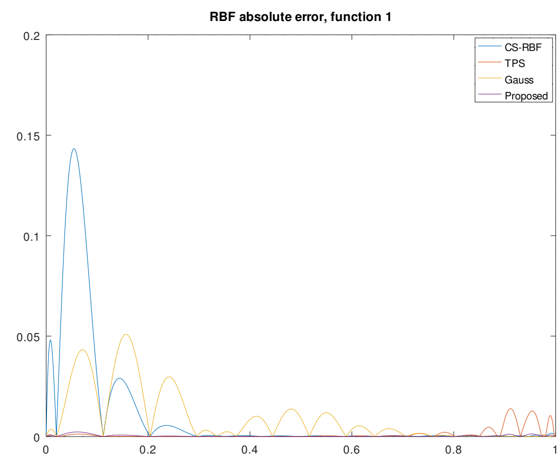

Fig. 9: Function 3 differences.

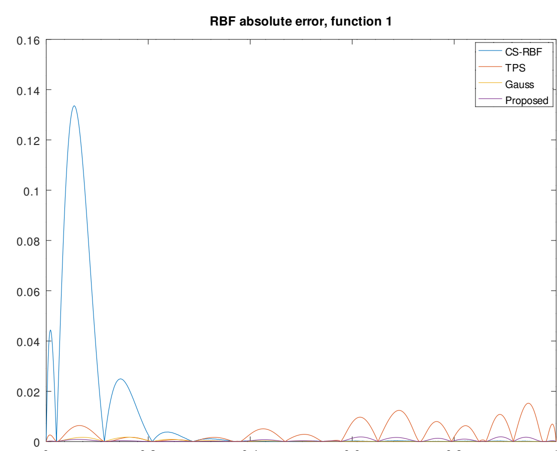

Fig. 10: Function 4 differences.

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Function number	Radial basis function			
	CS-RBF	TPS	Gauss	Proposed
1	$7.33 \cdot 10^{-2}$	$7.76 \cdot 10^{-3}$	$1.70 \cdot 10^{-3}$	$9.21 \cdot 10^{-4}$
2	$1.81 \cdot 10^{0}$	$5.24 \cdot 10^{-3}$	$5.19 \cdot 10^{-1}$	$6.29 \cdot 10^{-3}$
3	$1.44 \cdot 10^{-1}$	$1.39 \cdot 10^{-2}$	$5.09 \cdot 10^{-2}$	$2.30 \cdot 10^{-3}$
4	$1.33 \cdot 10^{-1}$	$1.52 \cdot 10^{-2}$	$1.76 \cdot 10^{-3}$	$1.89 \cdot 10^{-3}$

TABLE IV: Maximum absolute error.

Function number	Radial basis function			
	CS-RBF	TPS	Gauss	Proposed
1	$6.60 \cdot 10^{-19}$	$8.80 \cdot 10^{-6}$	$1.44 \cdot 10^{-12}$	$2.49 \cdot 10^{-9}$
2	$2.68 \cdot 10^{-18}$	$8.47 \cdot 10^{-5}$	$2.43 \cdot 10^{-12}$	$1.05 \cdot 10^{-7}$
3	$4.05 \cdot 10^{-18}$	$4.58 \cdot 10^{-6}$	$2.35 \cdot 10^{-12}$	$2.25 \cdot 10^{-9}$
4	$2.78 \cdot 10^{-18}$	$2.44 \cdot 10^{-6}$	$9.17 \cdot 10^{-14}$	$3.20 \cdot 10^{-10}$

TABLE V: Condition numbers.

tional points are not included in the approximation. (maximum error for the function 2, if CS-RBF is used).

The proposed RBF approximation method was also tested for a newly developed RBF. The experiments proved significant precision improvement of the final approximation over the TPS function.

VIII. Conclusion

In this contribution a novel approach for RBF approximation based on geometrical properties of a sampled curve (signal) is presented. Experiments proved advantages of the global functions over CS-RBFs are sensitive to the shape parameter selection and require more points for acceptable approximation in general.

The newly developed RBF function is better in the precision terms over the TPS function, however, the TPS function has a little bit conditionality of the RBF metrics.

The experiments also proved that the CS-RBFs require variable shape parameter which is significant result as CS-RBFs are used in many areas, e.g. solution of partial differential equations, etc. The adaptive shape parameter for CS-RBFs is to be explored in future.

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