International Journal of Mathematics and Computer Science, Vol.15, No.3, pp.769-777, ISSN 1814-0432, 2020

Scientific Computing and Computer Graphics with GPU: Application of Projective Geometry and Principle of Duality

V. Skala,[1, a)] S.A.A. Karim,[2, b)] and E.A.Kadir[3, c)]
[1)] *University of West Bohemia,*
CZ 301 00 Plzen Czech Republic
[2)] *Fundamental and Applied Sciences Department and Centre for Smart Grid Energy Research (CSMER),*
Institute of Autonomous System Universiti Teknologi PETRONAS Malaysia.
[3)] *Department of Informatics and Faculty of Engineering and Universitas Islam Riau,*
Jl. Kaharuddin Nasution and Marpoyan and Pekanbaru and Riau and Indonesia

Abstract. Geometric problems are usually solved in the Euclidean space by using the standard vector algebra techniques. In this study, principles of the projective geometry and geometric algebra will be introduced via a novel method that significantly simplifies the solution of geometrical problems. Also, it supports the GPU parallel computation application. Besides that, an application of the principle of duality leads to a simple solution of the dual problems. We show that, the equivalence of the extended cross-product (outer product) and the solution of the system of linear equations. This gives a direct impact to scientific computation, solution of geometrical problems, robotics, computer graphics algorithms and virtual reality via fast computation through GPU parallel systems. Some numerical and graphical results are presented.

GEOMETRIC ALGEBRA

The vector algebra (Gibbs algebra) used nowadays uses two basic operations on two vectors $\mathbf{a}, \mathbf{b}$ in E^n, i.e. the inner product (scalar product or dot product) $c = \mathbf{a} \cdot \mathbf{b}$, where c is a scalar value and outer product (the cross-product in E^3) $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$, where $\mathbf{c}$ is a bivector and has a different properties than a vector as it represents an oriented area in n-dimensional space, in general.

The Geometric Algebra (GA) uses a “new” product called Geometric product defined as:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{1}$$

where **ab** is a geometric product.

In the case of the n-dimensional space, vectors are defined as $\mathbf{a} = (a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n)$, $\mathbf{b} = (b_1\mathbf{e}_1 + \ldots + b_n\mathbf{e}_n)$ and the $\mathbf{e}_i$ vectors form orthonormal basis vectors in E^3 then we get:

1	0-vector (scalar)	$\mathbf{e}_{12}$, $\mathbf{e}_{23}$, $\mathbf{e}_{31}$	2-vectors (bivectors)
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3,$	1-vector (vectors)	$\mathbf{e}_{123}$	3-vector (pseudoscalar)

It can be easily proved that the following operations are valid, including an inverse of a vector.

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \qquad \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \qquad \mathbf{a}^{-1} = \mathbf{a}/||\mathbf{a}||^2 \tag{2}$$

It can be seen, that geometric algebra is *anti-commutative* and the “pseudoscalar” I in E^3 has the basis $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, i.e.

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \qquad \mathbf{e}_i\mathbf{e}_i = 1 \qquad \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = I \qquad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = q \tag{3}$$

where q is a scalar value.

In general, the geometric product is represented as:

$$\mathbf{ab} = \sum_{i,j=1}^{n,n} a_i\mathbf{e}_i b_j\mathbf{e}_j \qquad \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n,n} a_i\mathbf{e}_i b_i\mathbf{e}_i \tag{4}$$

[a)] Electronic mail: http://www.VaclavSkala.eu
[b)] Corresponding author: samsul_ariffin@utp.edu.my
[c)] Electronic mail: evizal@eng.uir.ac.id

$$\mathbf{a}\wedge\mathbf{b} = \sum_{i,j=1\&i\neq j}^{n,n} a_i\mathbf{e}_i b_j\mathbf{e}_j = \sum_{i,j=1,\&i>j}^{n} (a_ib_j - a_jb_i)\mathbf{e}_i\mathbf{e}_j \tag{5}$$

It is not a “friendly user” notation for a practical application and causes problems in practical implementations, especially due to anti-commutativity of the geometric product.

However, the geometric product can be easily represented by the tensor product, which can be represented by a matrix. As the homogeneous coordinates will be used in the following, the tensor product for the 4-dimensional case is presented:

$$\mathbf{ab} \underset{\text{repr}}{\Longleftrightarrow} \mathbf{ab}^T = \mathbf{a}\otimes\mathbf{b} = \mathbf{Q} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_1b_2 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_1b_3 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_1b_4 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix} = \mathbf{B}+\mathbf{U}+\mathbf{D} \tag{6}$$

where $\mathbf{B}+\mathbf{U}+\mathbf{D}$ are *B*ottom triangular, *U*pper triangular, *D*iagonal matrices, a_4, b_4 are the homogeneous coordinates, i.e. actually w_a, w_b (will be explained later), and the operator $\otimes$ means the anti-commutative tensor product.

PROJECTIVE EXTENSION AND PRINCIPLE OF DUALITY

Let us consider the projective extension of the Euclidean space and use of the homogeneous coordinates. Let us consider vectors $\mathbf{a} = [a_1, a_2, a_3 : a_4]^T$ and $\mathbf{b} = [b_1, b_2, b_3 : b_4]^T$, which represents actually vectors $(a_1/a_4, a_2/a_4, a_3/a_4)$ and $(b_1/b_4, b_2/b_4, b_3/b_4)$ in the E^3 space. It can be seen, that the diagonal of the matrix $\mathbf{Q}$ actually represents the inner product in the projective representation:

$$\mathbf{a}\cdot\mathbf{b} = [(a_1b_1 + a_2b_2 + a_3b_3) : a_4b_4]^T \triangleq \frac{a_1b_1 + a_2b_2 + a_3b_3}{a_4b_4} \tag{7}$$

where $\triangleq$ means projectively equivalent. The inner product actually represents trace $tr(\mathbf{Q})$ of the matrix $\mathbf{Q}$.

The outer product (the cross-product in the E^3 case) is then represented respecting anti-commutativity as:

$$\mathbf{a}\wedge\mathbf{b} \underset{\text{repr}}{\Longleftrightarrow} \sum_{i,j=1\&i>j}^{3,3} (a_ib_j\mathbf{e}_i\mathbf{e}_j - b_ia_j\mathbf{e}_i\mathbf{e}_j) = \sum_{i,j\&i>j}^{3,3} (a_ib_j - b_ia_j)\mathbf{e}_i\mathbf{e}_j \tag{8}$$

It should be noted, that the outer product can be used for a solution of a linear system of equations $\mathbf{Ax} = \mathbf{b}$ or $\mathbf{Ax} = \mathbf{0}$, too.

The principle of duality is an important principle, in general. Its application in geometry in connection with the implicit representation using projective geometry brings some new formulations or even new theorems. The duality principle for basic geometric entities and operators are presented by TAB.I and TAB.II.

TABLE I: Duality of geometric entities

Duality of geometric entities						
Point in E^2	$\underset{\text{DUAL}}{\Longleftrightarrow}$	Line in E^3		Point in E^3	$\underset{\text{DUAL}}{\Longleftrightarrow}$	Plane in E^3

TABLE II: Duality of operators

Duality of operators		
Union $\cup$	$\underset{\text{DUAL}}{\Longleftrightarrow}$	Intersection $\cap$

It means, that in the E^2 case a point is dual to a line and vice versa, intersection of two lines is dual to a union of two points, i.e. line given by two points; similarly for the E^3 case.

COMPUTATION WITH HOMOGENEOUS REPRESENTATION

The direct consequence of the principle of duality is that, the intersection point $\mathbf{x}$ of two lines $\mathbf{p}_1, \mathbf{p}_2$, resp. a line $\mathbf{p}$ passing two given points $\mathbf{x}_1, \mathbf{x}_2$, is given as:

$$\mathbf{x} = \mathbf{p}_1 \wedge \mathbf{p}_2 \underset{\text{DUAL}}{\Longleftrightarrow} \mathbf{p} = \mathbf{x}_1 \wedge \mathbf{x}_2 \tag{9}$$

where $\mathbf{p}_i = [a_i, b_i : c_i]^T$, $\mathbf{x} = [x, y : w]^T$ (w is the homogeneous coordinate), $i = 1, 2$; similarly in the dual case.

In the case of the E^3 space, a point is dual to a plane and vice versa. It means that the intersection point $\mathbf{x}$ of three planes ρ_1,ρ_2,ρ_3, resp. a plane ρ passing three given points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is given as:

$$\mathbf{x} = \rho_1 \wedge \rho_2 \wedge \rho_3 \underset{\text{DUAL}}{\Longleftrightarrow} \rho = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \tag{10}$$

where $\mathbf{x} = [x, y, z : w]^T$, $\rho_i = [a_i, b_i, c_i : d_i]^T$, $i = 1, 2, 3$.

It can be seen that the above formulae is equivalent to the "extended" cross-product, which in natively supported by GPU architecture. For an intersection computation, we get:

$$\mathbf{x} = \mathbf{p}_1 \wedge \mathbf{p}_2 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_w \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \qquad \mathbf{x} = \rho_1 \wedge \rho_2 \wedge \rho_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_w \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \tag{11}$$

Due to the principle of duality, a dual problem solution is given as:

$$\mathbf{p} = \mathbf{x}_1 \wedge \mathbf{x}_2 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_w \\ x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \end{bmatrix} \qquad \rho = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_w \\ x_1 & y_1 & z_1 & w_1 \\ x_1 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{bmatrix} \tag{12}$$

The above presented formulae prove the strength of the formal notation of the geometric algebra approach. Therefore, there is a natural question, what is the more convenient computation of the geometric product, as computation with the outer product, i.e. extended cross product, using basis vector approach is not simple.

Fortunately, the geometric product of ρ_1, ρ_2, resp. of $\mathbf{x}_1$ and $\mathbf{x}_2$ vectors using homogeneous coordinates given as anti-commutative tensor product is given as:

$\rho_1\rho_2$	a_2	b_2	c_2	d_2
a_1	a_1a_2	a_1b_2	a_1c_2	a_1d_2
b_1	b_1a_2	b_1b_2	b_1c_2	b_1d_2
c_1	c_1a_2	c_1b_2	c_1c_2	a_1d_2
d_1	d_1a_2	d_1b_2	d_1c_2	d_1d_2

$\mathbf{x}_1\mathbf{x}_2$	x_2	y_2	z_2	w_2
x_1	x_1x_2	x_1y_2	x_1z_2	x_1w_2
y_1	y_1x_2	y_1y_2	y_1z_2	y_1w_2
z_1	z_1x_2	z_1y_2	z_1z_2	x_1w_2
w_1	w_1x_2	w_1y_2	w_1z_2	w_1w_2

However, the question is how to compute a line $\mathbf{p} \in E^3$ given as an intersection of two planes ρ_1, ρ_2, which is dual to a line determination given by two points $\mathbf{x}_1$, $\mathbf{x}_2$ as those problems are dual.

The parametric solution can be easily obtained using standard Plücker coordinates, however computation and formula are complex and not easy to understand.

$$q(t) = \frac{\omega \times \mathbf{v}}{||\omega||^2} + \omega t \qquad \mathbf{L} = \mathbf{x}_1\mathbf{x}_2^T - \mathbf{x}_2\mathbf{x}_1^T \tag{13}$$

$$\omega = [l_{41}, l_{42}, l_{43}]^T \qquad \mathbf{v} = [l_{23}, l_{31}, l_{12}]^T \tag{14}$$

For the case of intersection of two planes the principle of duality can be applied directly.

However, using the geometric algebra, principle of duality and projective representation, we can directly write:

$$\mathbf{p} = \boldsymbol{\rho_1} \wedge \boldsymbol{\rho_2} \underset{\text{DUAL}}{\Longleftrightarrow} \mathbf{p} = \mathbf{x}_1 \wedge \mathbf{x}_2 \tag{15}$$

It can be seen that the formula given above keeps the duality in the final formulae, too.

From the formal point of view, the geometric product for the both cases is given as:

$$\rho_1\rho_2 \underset{\text{repr}}{\Longleftrightarrow} \rho_1 \otimes \rho_2 = \begin{bmatrix} a_1a_2 & a_1b_2 & a_1c_2 & a_1d_2 \\ b_1a_2 & b_1b_2 & b_1c_2 & b_1d_2 \\ c_1a_2 & c_1b_2 & c_1c_2 & c_1d_2 \\ d_1a_2 & d_1b_2 & d_1c_2 & d_1d_2 \end{bmatrix} \tag{16}$$

The dual problem formulation:

$$\mathbf{x}_1\mathbf{x}_2 \underset{\text{repr}}{\Longleftrightarrow} \mathbf{x}_1 \otimes \mathbf{x}_2 = \begin{bmatrix} x_1x_2 & x_1y_2 & x_1z_2 & x_1w_2 \\ y_1x_2 & y_1y_2 & y_1z_2 & y_1w_2 \\ z_1x_2 & z_1y_2 & z_1z_2 & z_1w_2 \\ w_1x_2 & w_1y_2 & w_1z_2 & w_1w_2 \end{bmatrix} \tag{17}$$

It means that we have computation of the Plücker coordinates for the both cases, i.e. for computation of a line $\mathbf{p} = \rho_1 \wedge \rho_2$ or $\mathbf{p} = \mathbf{x}_1 \wedge \mathbf{x}_2$ is given as a union of two points in E^3 and as an intersection of two planes in E^3 using the projective representation and the principle of duality. It should be noted that the given approach offers: significant simplification of computation of the Plücker coordinates as it is simple and easy to derive and explain, uses vector-vector operations, which is especially convenient for SSE and GPU application one code sequence for the both cases.

As the Plücker coordinates are also in mechanical engineering applications, especially in robotics due to its simple displacement and momentum specifications, and in other fields simple explanation and derivation is another very important argument for GA approach application.

SOLUTION OF LINEAR SYSTEM OF EQUATIONS

A solution of a linear system of equations is a part of the linear algebra and used in many computational systems. It should be noted, that linear equations $\mathbf{Ax} = \mathbf{b}$ can be transformed to an implicit the homogeneous system, i.e. to the form $\mathbf{B}\xi = \mathbf{0}$, where $\mathbf{B} = [\mathbf{A}|-\mathbf{b}]$, $\xi = [\xi_1, ..., \xi_n : \xi_w]^T$, $x_i = \xi_i / \xi_w$, $i = 1, ..., n$.

As the solution of a linear system of equations is equivalent to the outer product (generalized cross-vector) of vectors formed by rows of the matrix $\mathbf{B}$, the solution of the system is defined as:

$$\xi = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge ... \wedge \mathbf{a}_n \qquad\qquad [\mathbf{A}| - \mathbf{b}]\xi = 0 \tag{18}$$

which is equivalent to a solution of the linear system of equations:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{19}$$

It a very important result as a solution of a linear system of equations is formally the same for systems for the both cases, i.e. $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$. As the solution is formally determined, the formal linear operators can be used for further symbolic processing using formula manipulation, as the geometry algebra is multilinear. Even more, it is capable to handle more complex objects generally in the d-dimensional space, i.e. oriented surfaces, volumes etc. Therefore, it is possible to use the Functional analysis approach: "Let L is a linear operator, then the following operation is valid....". As there are many linear operators like derivation, integration, Laplace transform etc., there is a huge potential of applications of those to the formal solution of the linear system of equations, i.e. $L(\xi)$. However, it is necessary to respect, that in the case of projective representation a specific care is to be taken for deriving rules for derivation etc., as actually a fraction is to be process and similarly for other operators.

CONCLUSION

This contribution briefly presents geometry algebra, which is not generally known and used. However, it offers simple and efficient solutions to many computational problems, if combined with the principle of duality and projective notation.

As the result of this contribution a new formulation of the Plücker coordinates, often used in mechanical engineering and robotics, is given. As the operations are based on standard linear algebra formalism it is simple to use. The presented approach supports direct GPU application with a potential of significant speed-up and parallelism. Also, the approach is applicable to d-dimensional problem solutions, as the geometric algebra is multidimensional.

ACKNOWLEDGMENTS

The authors would like to thank their colleagues and students at the University of West Bohemia and Universiti Teknologi PETRONAS for their discussions and suggestions. Thanks belong also to anonymous reviewers for their valuable comments and hints provided. This research was supported by the Czech Science Foundation (GACR) GA 17 05534S and Universiti Teklnologi PETRONAS (UTP) through YUTP:0153AA-H24 and Universitas Islam Riau (UIR), Pekan baru, Indonesia and Universiti Teknologi PETRONAS (UTP), Malaysia through **International Collaborative Research Funding (ICRF): 015ME0-037**.

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