Geometry algebra and conditionality of linear system of equations

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1 Introduction

A new approach to the matrix conditionality and the solvability of the linear systems of equations is presented. It is based on the application of the *geometric algebra* with the projective space representation using homogeneous coordinates representation. There are two main groups:

- non-homogeneous systems of linear equations, i.e. $\mathbf{Ax} = \mathbf{b}$
- homogeneous system of equations, i.e. $\mathbf{Ax} = \mathbf{0}$

Using the *principle of duality* and *projective extension of the Euclidean space* the first type of the linear system, i.e. $\mathbf{Ax} = \mathbf{b}$, can be easily transformed to the second type, i.e. $\mathbf{Ax} = \mathbf{0}$. The *geometric algebra* offers more general formalism, which can be used for a better understanding of the linear system of equations properties and behavior.

1.1 Geometric algebra

The *Geometric Algebra* (GA) uses a "new" product called *geometric product* defined as:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{1}$$

where $\mathbf{ab}$ is the new entity. It should be noted, that it is a "bundle" of objects with different dimensionalities and properties, in general. In the case of the n-dimensional space, the vectors are defined as $\mathbf{a} = (a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n)$, $\mathbf{b} = (b_1\mathbf{e}_1 + \ldots + b_n\mathbf{e}_n)$ and the $\mathbf{e}_i$ vectors form orthonormal vector basis in E^n. In the E^3 case, the following objects can be used in geometric algebra: [5]:

1	0-vector (scalar)	$\mathbf{e}_{12}$, $\mathbf{e}_{23}$, $\mathbf{e}_{31}$	2-vectors (bivectors)
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3,$	1-vector (vectors)	$\mathbf{e}_{123}$	3-vector (pseudoscalar)

The significant advantage of the geometric algebra is, that it is more general that than the Gibbs algebra and can handle all objects with dimensionality up to n. The geometry algebra uses the following operations, including the inverse of a vector.

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \qquad \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \qquad \mathbf{a}^{-1} = \mathbf{a}/||\mathbf{a}||^2 \tag{2}$$

It should be noted, that geometric algebra is *anti-commutative* and the "pseudoscalar" I in the E^3case has the basis $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ (briefly as $\mathbf{e}_{123}$), i.e.

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \qquad \mathbf{e}_i\mathbf{e}_i = 1 \qquad \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = I \qquad \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = q \tag{3}$$

where q is a scalar value (actually a pseudoscalar).

2 Solution of linear systems of equations

The linear system of equations $\mathbf{Ax} = \mathbf{b}$ can be transformed to the homogeneous system of linear equations, i.e. to the form $\mathbf{D}\xi = \mathbf{0}$, where $\mathbf{D} = [\mathbf{A}|-\mathbf{b}]$, $\xi = [\xi_1, ..., \xi_n : \xi_w]^T$, $x_i = \xi_i \,/\, \xi_w$, $i = 1, ..., n$. If $\xi_w \mapsto 0$ then the solution is in infinity and the vector $(\xi_1, ..., \xi_n)$ gives the "direction", only.

As the solution of a linear system of equations is equivalent to the outer product (generalized cross-vector) of vectors formed by rows of the matrix $\mathbf{D}$, the solution of the system $\mathbf{D}\xi = \mathbf{0}$ is defined as:

$$\xi = \mathbf{d}_1 \wedge \mathbf{d}_2 \wedge ... \wedge \mathbf{d}_n \qquad \mathbf{D}\xi = 0 \qquad , i.e. \qquad [\mathbf{A}| - \mathbf{b}]\xi = 0 \tag{4}$$

where: $\mathbf{d}_i$ is the i-th row of the matrix $\mathbf{D}$, i.e. $\mathbf{d}_i = (a_{i1}, ..., a_{in}, -b_i)$, $i = 1, ..., n$. The application of the projective extension of the Euclidean space enables us to transform the non-homogeneous system of linear equations $\mathbf{Ax} = \mathbf{b}$ to the homogeneous linear system $\mathbf{D}\xi = \mathbf{0}$, i.e.:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \underset{\text{conversion}}{\Longleftrightarrow} \quad \begin{bmatrix} a_{11} & \cdots & a_{1n} & -b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & -b_n \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \\ \xi_w \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5}$$

It is an important result as a solution of a linear system of equations is formally the same for both types, i.e. homogeneous linear systems $\mathbf{Ax} = \mathbf{0}$ and non-homogeneous systems $\mathbf{Ax} = \mathbf{b}$.

2.1 Angular criterion

Both types of the linear systems of equations, i.e. $\mathbf{Ax} = \mathbf{b}$ ($\mathbf{A}$ is $n \times n$) and $\mathbf{Ax} = \mathbf{0}$ ($\mathbf{A}$ is $(n+1) \times n$), actually have the same form $\mathbf{Ax} = \mathbf{0}$ ($\mathbf{A}$ is $(n+1) \times n$), now, if the projective representation is used. Therefore, it is possible to show the differences between the matrix conditionality and conditionality

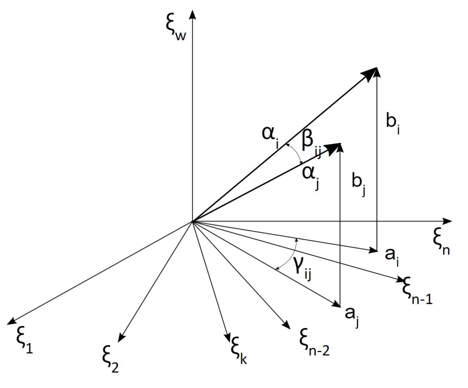

Figure 1: Difference between matrix and linear system conditionality

(solvability) of a linear system of equations, see Fig.1.

The eigenvalues are usually used and the ratio $rat_\lambda = |\lambda_{max}|/|\lambda_{min}|$ & $\lambda \in C$ is mostly used as a criterion. If the ration rat_λ is high, the matrix is said to be ill-conditioned, especially in the case of large data with a large span of data. There are two cases, which are needed to be taken into consideration:

- non-homogeneous systems of linear equations, i.e. $\mathbf{Ax} = \mathbf{b}$. In this case, the matrix conditionality is considered as a criterion for the solvability of the linear system of equations. It depends on the matrix $\mathbf{A}$ properties, i.e. on eigenvalues.

Table 1: Conditionality of modified the Hilbert matrix: Experimental results (*with Octave warnings)

N	cond($\mathbf{H}_{orig}$)	cond($\mathbf{H}_{new}$)		N	cond($\mathbf{H}_{orig}$)	cond($\mathbf{H}_{new}$)
3	5.2406e+02	2.5523e+02		7	4.7537e+08	1.4341e+08
4	1.5514e+04	6.0076e+03		8	1.5258e+10	6.0076e+03
5	4.7661e+05	1.6099e+05		9	4.9315e+11	1.3736e+11
6	1.4951e+07	5.0947e+06		10	1.6024e+13	4.1485e+12
				20	1.6024e+13*	4.1485e+12

$$\begin{bmatrix} 10^2 & 0 & 0 \\ 0 & 10^0 & 0 \\ 0 & 0 & 10^{-2} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_3 \end{bmatrix} \tag{6}$$

A conditionality number $\kappa(\mathbf{A}) = |\lambda_{max}|/|\lambda_{min}|$ is usually used as the solvability criterion. In the case of the Eq.6, the matrix conditionality is $\kappa(\mathbf{A}) = 10^2/10^{-2} = 10^4$. However, if the 1^{st} row is multiplied by 10^{-2} and the 3^{rd} row is multiplied by 10^2, then the conditionality is $\kappa(\mathbf{A}) = 1$.

- a homogeneous system of equations $\mathbf{Ax} = \mathbf{0}$, when the system of linear equations $\mathbf{Ax} = \mathbf{b}$ is expressed in the projective space. In this case, the vector $\mathbf{b}$ is taken into account and bivector area and bivector angles properties can be used for solvability evaluation.

The only *angular* criterion is invariant to the row multiplications, while only the column multiplication changes angles of the bivectors. There are several significant consequences:

- the solvability of a linear system of equations can be improved by the column multiplications, only, if unlimited precision is considered. Therefore, the matrix-based pre-conditioners might not solve the solvability problems and might introduce additional numerical problems.
- the precision of computation is significantly influenced by addition and subtraction operations, as the exponents must be the same for those operations with mantissa. Also, the multiplication and division operations using exponent change by $2^{\pm k}$ should be preferred.

2.2 Preconditioning simplified

There are several methods used to improve the ratio $\kappa(\mathbf{A}) = |\lambda_{max}|/|\lambda_{min}|$ of the matrix $\mathbf{A}$ of the linear system, e.g. matrix eigenvalues shifting or preconditioning [1] [2]. The preconditioning is usually based on solving a linear system $\mathbf{Ax} = \mathbf{0}$:

$$\mathbf{PAS}\ \mathbf{S}^{-1}\mathbf{x} = \mathbf{Pb} \tag{7}$$

where $\mathbf{P}$ is a matrix, which can cover complicated computation, including Fourier transform. The inverse operation, i.e. $\mathbf{P}$, is computationally very expensive as it is of $O(n^3)$ complexity. Therefore, they are not easily applicable for large systems of linear equations used nowadays. There are methods based on incomplete factorization, etc., which might be used [3]. The proposed matrix conditionality improvement method requires only the diagonal matrices values $\mathbf{P}$ and $\mathbf{S}$, i.e. multiplicative coefficients $p_i \neq 0$, $s_j \neq 0$, which have to be optimized. This is a significant reduction of computational complexity, as it decreases the cost of finding sub-optimal p_i, s_j values. The proposed approach was tested on the Hilbert's matrix as conditionality can be estimated as $\kappa(\mathbf{H_n}) \simeq e^{3.5n}$. The experimental results of the original conditionality $\kappa(\mathbf{H}_{orig})$ and conditionality using the proposed method $\kappa(\mathbf{H}_{new})$ are presented in Tab.1.

Table 2: Conditionality of modified the Hilbert matrix: Experimental results (*with Octave warnings)

N	3	4	5	6	7
$\kappa_{rat}(\mathbf{H}_{orig})$	0.54464	0.39282	0.31451	0.26573	0.23195
$\kappa_{rat}(\mathbf{H}_{new})$	0.98348	0.97740	0.98173	0.96283	0.87961
N	8	9	10		20
$\kappa_{rat}(\mathbf{H}_{orig})$	0.20694	0.18755	0.17199	$\cdots$	0.09917*
$\kappa_{rat}(\mathbf{H}_{new})$	0.92500	0.96435	0.96322	$\cdots$	0.74701*

The experiments proved, that the conditionality cond($\mathbf{H}_{new}$) of the modified matrix using the proposed approach was decreased by more than half of the magnitude for higher values of n, see Tab.1. This is consistent with the recently obtained results [4], where the inverse Hilbert matrix computation using the modified Gauss elimination without division operation was analyzed.

The Hilbert matrix conditionality improvement also improved the angular criterion based on maximizing the ratio $\kappa_{rat}(\mathbf{H})$ defined as:

$$\kappa_{rat}(\mathbf{H}) = \frac{\cos\gamma_{min}}{cos\gamma_{max}} \qquad \kappa_{rat}(\mathbf{H}) = \frac{\cos\beta_{min}}{cos\beta_{max}} \tag{8}$$

It says, how the angles $\cos\gamma_{ij}$, formed by the vectors $\mathbf{a}_{ij}$ of the bivectors are similar, see Fig.1. It means, that if the ratio $\kappa_{rat}(\mathbf{A}) \simeq 1$ the angles of all bivectors are nearly equal. In the case of conditionality assessment of the linear system of equations $\mathbf{Ax} = \mathbf{0}$, the angles β_{ij}, formed by the angels α_{ij} have to be taken into account, see Fig.1. The results presented in Tab.2 reflects the improvement of the Hilbert matrix by proposed approach using the diagonal matrices $\mathbf{P}$ and $\mathbf{S}$ used as the multipliers.

3 Conclusion

The advantage of the angular criterion is that it is common for the conditionality evaluation of the matrix and of the linear system of equations. It should be noted, that this conditionality assessment method gives different values of conditionality of those two different cases, as in the first case only the matrix is evaluated, while in the second one the value of the $\mathbf{b}$ in the $\mathbf{Ax} = \mathbf{b}$ is taken into account.

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