Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

Simple and Efficient Acceleration of the Smallest Enclosing Ball for Large Data Sets in E^2: Analysis and Comparative Results ⋆

Vaclav Skala[0000−0001−8886−4281], Matej Cerny, and Josef Yassin Saleh

University of West Bohemia, Faculty of Applied Sciences
Dept. of Computer Science and Engineering
Pilsen, CZ 301 00, Czech Republic
skala@kiv.zcu.cz matcerny@kiv.zcu.cz salehj@kiv.zcu.cz
www.VaclavSkala.eu

Abstract. Finding the smallest enclosing circle of the given points in E^2 is a seemingly simple problem. However, already proposed algorithms have high memory requirements or require special solutions due to the great recursion depth or high computational complexity unacceptable for large data sets, etc. This paper presents a simple and efficient method with speed-up over 100 times based on processed data reduction. It is based on efficient preprocessing, which significantly reduces points used in final processing It also significantly reduces the depth of recursion and memory requirements, which is a limiting factor for large data processing. The proposed algorithm is easy to implement and it is extensible to the E^3 case, too. The proposed algorithm was tested for up to 10^9 of points using the Halton's and "Salt and Pepper" distributions.

Keywords: Smallest enclosing circle · smallest enclosing ball · algorithm complexity · preprocessing · convex hull · convex hull diameter.

1 Introduction

Algorithms for finding the smallest enclosing circle in the E^2 case, or the enclosing ball in the E^k general case, have been studied for a long time and many algorithms have been published with many modifications. Sylvester[62] made the first problem formulation in 1857 and later by others, see Elzinga[10]. Several algorithms have been published, e.g. Megiddo's algorithm[31] with an overview of some other interesting solutions, Ritter[46], etc. A brief introduction to the problem is available at WiKi[71][72].

An interesting approach was published by Welzl[67] in 1991. It is a "brute force" recursive algorithm with a random selection of points. It leads to a significant speed-up due to random point selection. However, it is not directly usable for large data sets due to the very deep recursion calls.

⋆ Research supported by the University of West Bohemia - Institutional research support No.1311.

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

Unfortunately, the originally proposed Welzl's algorithm is partially incorrect and Matoušek, Sharir, Welzl's published the corrected version, as the MSW algorithm[30] (the code available on WiKi[72]), see[70][72].

Algorithm 1 MSW - Matousek, Sharir, Welzl's algorithm

Require: Finite sets P and R of points in the plane $|R| = 3$
Ensure: Minimal disk enclosing $P \cup R$
  if P is empty **then**
    return trivial(R)
  end if
  choose p in P ▷ randomly and uniformly
  D := msw($P - \{p\}$, R)
  if p is in P **then**
    return(D)
  end if
  q = nonbase($R \cup \{p\}$)
    ▷ Welzl's algorithm for 4 points could be used to find what would not be in R
  return MSW ($P - \{p\} \cup \{q\}$, $R \cup \{p\} - \{q\}$)

It should be noted, that there is no significant difference between the original Welzl's and MSW algorithms as far as the timing is concerned.

2 Proposed preprocessing

The smallest enclosing center problem is closely related to the diameter of the convex hull problem. A simple algorithm with the $O_{exp}(N)$ complexity for finding a diameter of the convex hull of points using preprocessing was published by Skala[53][55][58]. The algorithm based on polar space subdivision was introduced in Skala,Smolik,Majdisova[59] and extended in Skala,Majdisova,Smolik[57] for the E^3 case.

2.1 Convex hull diameter estimation

It is based on a simple idea. The AABB points and points closest to the AABB corners form a convex hull. The maximum distance d defines an estimation of the convex hull diameter, i.e. radius r. Then the given points Ω are split to $\Omega_0, \ldots, \Omega_4$, see Fig.1.

It can be seen that the Ω_0 points cannot contribute to the convex hull diameter. Then points of Ω_1 and Ω_3 are processed and the value r is updated. Similarly, updates of the radius r are made after Ω_2 and Ω_4, Ω_1 and Ω_2, Ω_2 and Ω_3, Ω_3 and Ω_4, Ω_4 and Ω_1. This leads to significant reduction of points that could form the final convex hull. Then the final diameter of the convex hull is computed; see Skala[53][56][58] for details.

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

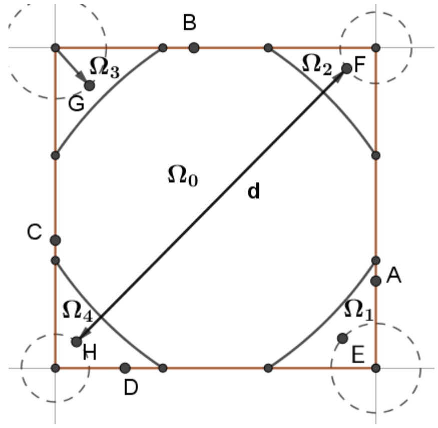


Fig. 1: Maximum distance estimation

Other efficient algorithms for finding the convex hull of points in the E^2 case were published in Skala[55][59] and for the convex hull in E^3 case was described in Skala[57]. This preprocessing leads to significant speed up, see Skala[58] for details. However, the polar subdivision used in Smolik[61] is quite complex to implement.

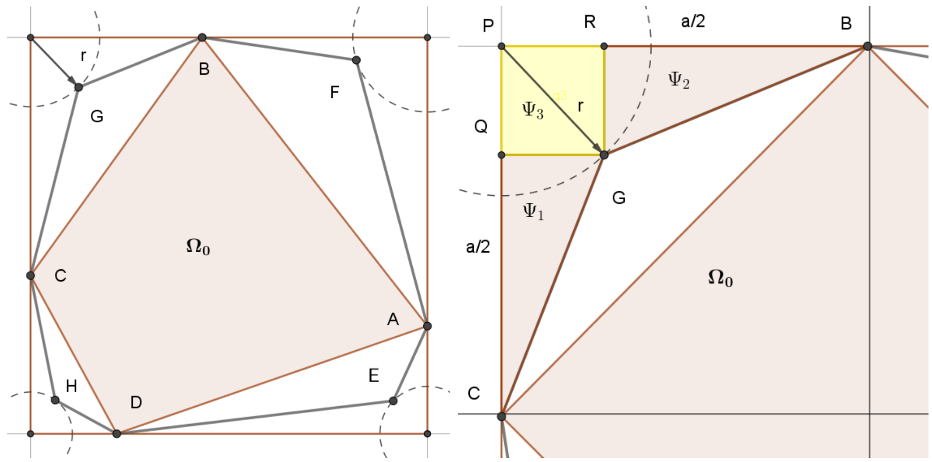
The Welzl's recursive algorithm is based on the "brute force" approach actually, but the randomized point heuristic selection use leads to the $O_{exp}(N)$ expected complexity, where N is a number of points. Unfortunately, it leads to deep recursive calls, which is a very limiting factor for large data processing.

2.2 Theoretical analysis

The simplest acceleration of the smallest enclosing circle algorithm is to find points that form the Axis Aligned Bounding Box (AABB), i.e. points A, B, C, D. The worst case is when the AABB is a square and the points A, B, C, D are at the middle of the edges, see Fig.2a. In this case, all points inside of the area Ω_0 can be removed from the future processing. However, if points closest to the AABB corners are found, i.e. points E, F, G, H, all points inside of the convex polygon $A, \ldots, H$ can be removed. The points E, F, G, H are on an expected distance r from corners and the radius r decreases with the number N of the given points.

The Fig.2a presents a general case with a rectangular area. The closest point to a corner of the AABB lies on a circle with the expected radius r. It should be noted that only $\frac{1}{4}$ of the circle area is inside of the AABB. The radius r depends on the number of the given points N. If the regular orthogonal distribution of points in the E^2 case forms a mesh of $\sqrt{N} \times \sqrt{N}$ points. However, in the following a uniform distribution of points is expected and the "Salt and Pepper" Chen[5] and Halton's[71] and distributions were used in experiments.

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52



(a) Convex hull estimation (b) Corner area

Fig. 2: Axis Aligned Bounding Box and convex polygon

Let P_0 is the area of the square with edges of the length a. The P_1 is the area of the circular sector of the radius r, see Fig.2a, containing just one point.

$$\frac{1}{N} = \frac{P_1}{P_0} \qquad pN = 1 \qquad p = \frac{P_1}{P_0} \qquad N\frac{\frac{1}{4}\pi r^2}{a^2} = 1 \qquad r^2 = \frac{4a^2}{\pi N} \tag{1}$$

Then in the E^2 case the expected radius r can be estimated as:

$$r = \frac{2a}{\sqrt{\pi}\sqrt{N}} = \frac{2}{\sqrt{\pi}}\frac{a}{\sqrt{N}} \approx 1.1284\frac{a}{\sqrt{N}} \tag{2}$$

The sizes of the corner's areas $\Psi_1 + \Psi_2 + \Psi_3$ can be estimated as:

$$P_2 = \frac{a}{2}\frac{r}{\sqrt{N}} \tag{3}$$

It means, that the P_2 area decreases with the value N significantly. If the total number of points is $N = 10^8$, i.e. $\sqrt{N} = 10^4$, then

$$r \approx 1.1284\frac{a}{10^4} = 0.00011284\ a \qquad P_2 \approx \frac{0.00011284}{2\sqrt{N}}\ a^2 \tag{4}$$

It can be seen, that the estimated point reduction is very high. In the E^3 case, the number of points in the given set is N and only $\frac{1}{8}$ of the corner ball volume are inside of the AABB. The expected radius r can be estimated as:

$$pN = 1 \qquad p = \frac{V_1}{V_0} \qquad N\frac{\frac{1}{8}\frac{4}{3}\pi r^3}{a^3} = 1 \qquad r^3 = \frac{6a^3}{\pi\sqrt[3]{N}} \tag{5}$$

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

In the E^3 case, the expected radius r can be estimated as:

$$r = \frac{\sqrt[3]{6}\ a}{\sqrt[3]{\pi}\sqrt[3]{N}} \approx 1.2407 \frac{a}{\sqrt[3]{N}} \tag{6}$$

This gives some estimation of the efficiency of the proposed preprocesing.

2.3 Implementation notes

The algorithm for finding the convex polygon $A, \dots, H$ consists of four passes with $O(N)$ complexity (N is the total number of points). However, it should be noted that after the second step a significant fraction of points is discarded already.

Algorithm 2 Smallest enclosing circle algorithm

Require: All given points in the $\boldsymbol{\Omega}$ set
Ensure: The smallest enclosing circle

FIND the points forming the AABB, i.e. $A, \dots, D$.
SPLIT the points into four areas half-planes based on their position relative to the line segments formed by the points $A, \dots, D$.
 ▷ points that do not fit into any of the half-planes, i.e. quad
 ▷ points inside of $ABCD$ quad, are promptly discarded
FIND points $E, \dots, H$. ▷ distance of the points is measured only to the corner
 ▷ of their respective half-plane
REMOVE the remaining points inside of the convex polygon $A, \dots, H$
 ▷ information about point's half-plane is used to reduce further testing
 ▷ the points in $\boldsymbol{\Omega_0}$ are removed as they cannot influence the final smallest ball
CALL the Welzl's algorithm [MSW] for the remaining points ▷ see Fig.5a

This approach of discarding the significant fraction of points at the beginning has proven to be superior to the simple point-in-polygon test, as such test needs to find the whole polygon first, which, among other things, requires measuring distance to all four AABB corners per point. [1]
Further reduction of read/write operations can be achieved by using separate data structure for storing the index of a region Ω_i for a given point.

3 Experimental results

The proposed modification of the Welzl's algorithm was tested for a large number of points (up to $N > 4 * 10^8$ points) in the E^2 case. The Halton's and "Salt and Pepper" distributions were used for experiments. Experiments proved the following expected properties:

[1] Also, instead of computing the distance between points d, the $\sqrt{d^2}$ should not be used and d^2 can be used for distance comparisons Skala[52][54]. Same idea can be applied to the radius of a circumscribed circle in Welzl's algorithm.

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

– the proposed smallest enclosing circle with preprocessing algorithm is of the $O_{exp}(N)$ time complexity even for large data sets (the Halton's and "Salt and Pepper" distributions used),
– timing and significant speed up due to preprocessing, see Fig.3a and Fig.3b respectively,
– the reduction ratio grows with the number of points $O\sqrt{N}$, see Fig.5a,
– the relative processing time $time/N$ is nearly constant for $N \geq 10^4$, see Fig.4b,
– significant decrease in the recursion depth as the result of preprocessing, which in turn leads to higher memory efficiency.

Implementation was done in C++, x64, compiler MSVC, Windows 10, 16GB RAM, Intel i7-10750H, 2.60GHz, 6 Cores CPU.
One of the main advantages of the proposed preprocessing is the significant reduction of the recursion depth. It can be seen that the depth recursion required is nearly 10^4 less if the proposed preprocessing is used, see Fig.5b for $N = 10^8$ points. There is also a direct influence on the computational time required.

4 Conclusion

The proposed algorithm for the acceleration of finding the smallest enclosing ball is simple, fast, robust and easy to implement. The significant advantages over recent solutions are:

– significant speed-up up to 10^2 times and it grows with the number of points,
– significant reduction of the depth of recursion, which is a limiting factor of the original algorithm for large data sets processing,
– significant data reduction before the final Welzl's (MSW) algorithm use,
– simple extensibility of the preprocessing algorithm to the E^3 case,
– better memory management, caching, during data processing,

The Halton's and "Salt and Pepper" distributions were used and the experimental results proved the speed-up expected. However, there is a potential for additional speed-up using SIMD instructions or GPU, use of the more advanced algorithms, e.g. the $O(\lg N)$ or $O(1)$ point in the convex polygon algorithms Skala[51], circumscribed sphere algorithm Skala[54]. [2]

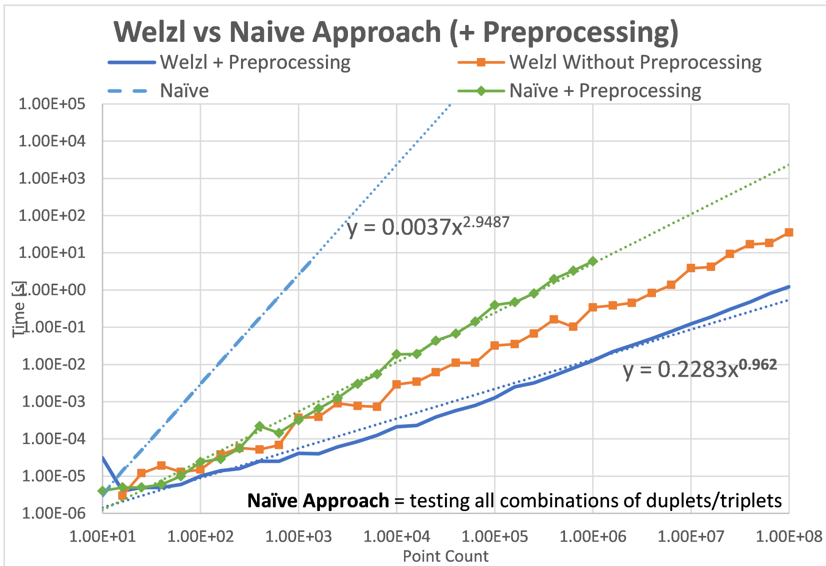
Acknowledgment

The authors would like to thank to colleagues at the Shandong University in Jinan (China) and University of West Bohemia in Pilsen (Czech Rep.) for stimulating of this work and discussions made. Thanks belong also to the anonymous reviewers for their critical comments, hints provided and for unknown relevant references. [3]

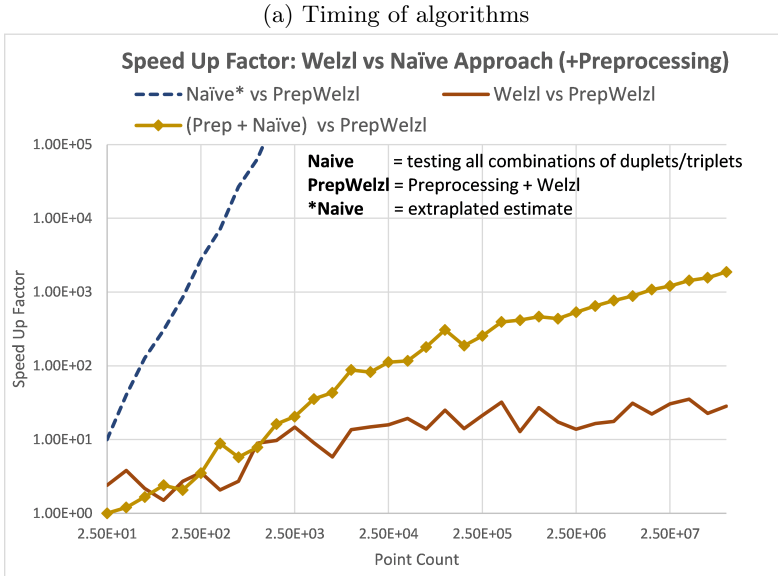
[2] Responsibilities: Algorithm design and analysis, manuscript preparation - V.Skala, Experimental implementation and verification - M.Cerny and J.Y. Saleh

[3] SIMD version using Intel's intrinsics AVX-2 has been tested and led to an additional roughly 20% performance gain.

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52



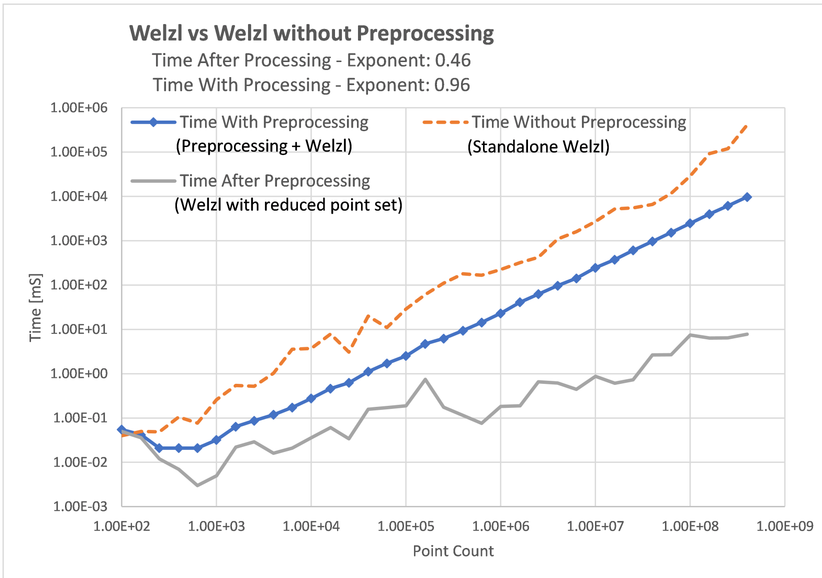
(a) Timing of algorithms



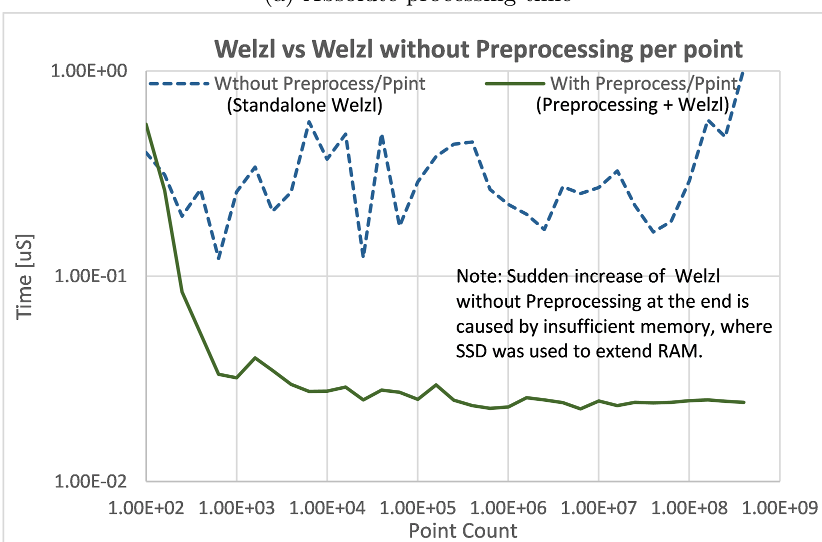
(b) The corner area influence

Fig. 3: Timing and corner areas influence

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52



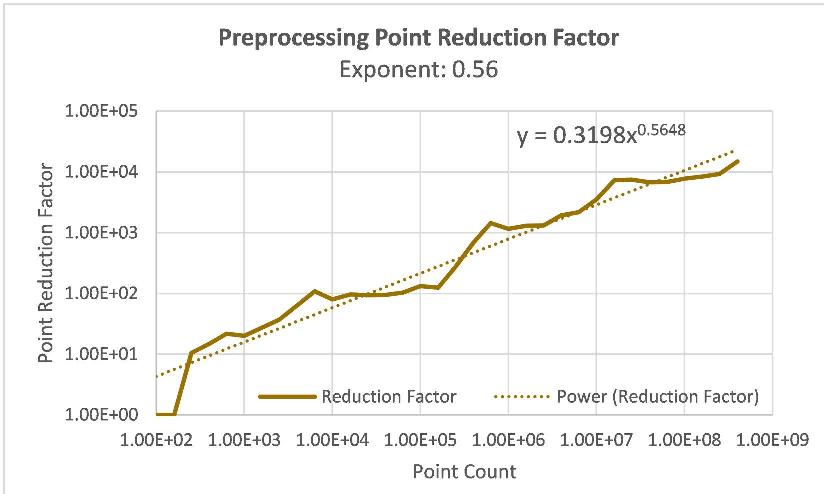
(a) Absolute processing time



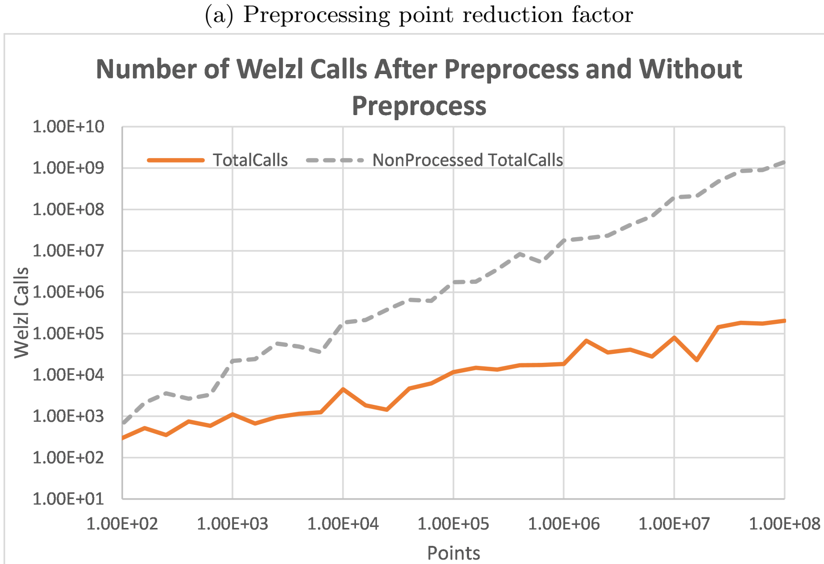
(b) Relative processing time, i.e. $\frac{time}{N}$

Fig. 4: Absolute and relative processing times

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52



(a) Preprocessing point reduction factor



(b) Comparison of the recursive depth

Fig. 5: Reduction and recursive depth

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

Appendix

This appendix presents related papers to the Smallest Enclosing Ball problem.

Agarwal[1], Cavaleiro[2][3], Cazals[4], Chen[5], Drager[6], Edelsbrunner[7], Efrat[8][9], Elzinga[10], wiki:[71][70][72], Fischer[12][13][11], Friedman[14], Gaertner[15], Gao[16], Goaoc[17], Har-Peled[18], Jiang[20][19], Kallberg[21], Karmakar[22][23], Krivosija[24], Larsson[25], Li[26], Liu[27], Martinetz[28], Martyn[29], Matousek[30], Megiddo[31], Mordukhovich[32], Mukherjee[33], Munteanu[34], Nam[35], Nielsen[38][39][40], Nielsen[41][42][37][36], Nock[43], Pan[44], Pronzato[45], Ritter[46], Saha[47], Shen[48], Shenmaier[49], Shi[50], Skyum[60], Smolik[61], Sylvester[62], Tao[63], Wang[64][65], Wei[66], Welzl[67][68][69], Xu[74][73], Yildirim[75], Zhou[76][77][78].

References

1. Agarwal, P., Ben Avraham, R., Sharir, M.: The 2-center problem in three dimensions. Computational Geometry: Theory and Applications **46**(6), 734–746 (2013). https://doi.org/10.1016/j.comgeo.2012.11.005
2. Cavaleiro, M., Alizadeh, F.: A faster dual algorithm for the euclidean minimum covering ball problem. Annals of Operations Research (2020). https://doi.org/10.1007/s10479-018-3123-5
3. Cavaleiro, M., Alizadeh, F.: A dual simplex-type algorithm for the smallest enclosing ball of balls. Computational Optimization and Applications **79**(3), 767–787 (2021). https://doi.org/10.1007/s10589-021-00283-6
4. Cazals, F., Dreyfus, T., Sachdeva, S., Shah, N.: Greedy geometric algorithms for collection of balls, with applications to geometric approximation and molecular coarse-graining. Computer Graphics Forum **33**(6), 1–17 (2014). https://doi.org/10.1111/cgf.12270
5. Chen, Q.Q., Hung, M.H., Zou, F.: Effective and adaptive algorithm for pepper-and-salt noise removal. IET Image Processing **11**(9), 709–716 (2017). https://doi.org/10.1049/iet-ipr.2016.0692
6. Drager, L., Lee, J., Martin, C.: On the geometry of the smallest circle enclosing a finite set of points. Journal of the Franklin Institute **344**(7), 929–940 (2007). https://doi.org/10.1016/j.jfranklin.2007.01.003
7. Edelsbrunner, H., Virk, Z., Wagner, H.: Smallest enclosing spheres and chernoff points in bregman geometry. Leibniz International Proceedings in Informatics, LIPIcs **99**, 351–3513 (2018). https://doi.org/10.4230/LIPIcs.SoCG.2018.35
8. Efrat, A., Sharir, M., Ziv, A.: Computing the smallest k-enclosing circle and related problems. Lecture Notes in Computer Science **709 LNCS**, 325–336 (1993). https://doi.org/10.1007/3-540-57155-8_259
9. Efrat, A., Sharir, M., Ziv, A.: Computing the smallest k-enclosing circle and related problems. Computational Geometry: Theory and Applications **4**(3), 119–136 (1994). https://doi.org/10.1016/0925-7721(94)90003-5
10. Elzinga, D.J., Hearn, D.W.: The minimum covering sphere problem. Manage. Sci. **19**(1), 96–104 (sep 1972). https://doi.org/10.1287/mnsc.19.1.96
11. Fischer, K., Gartner, B.: The smallest enclosing ball of balls: Combinatorial structure and algorithms. International Journal of Computational Geometry and Applications **14**(4-5), 341–378 (2004). https://doi.org/10.1142/s0218195904001500

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

12. Fischer, K., Gärtner, B.: The smallest enclosing ball of balls: Combinatorial structure and algorithms. Proceedings of the Annual Symposium on Computational Geometry pp. 292–301 (2003)
13. Fischer, K., Gärtner, B., Kutz, M.: Fast smallest-enclosing-ball computation in high dimensions. Lecture Notes in Computer Science **2832**, 630–641 (2003). https://doi.org/10.1007/978-3-540-39658-1_57
14. Friedman, F., Stroudsburg, E.: Minimal enclosing circle and two and three point partitions of a plane. In: In Proc.International Conference on Scientific Computing (2006)
15. Gaertner, B.: Fast and robust smallest enclosing balls. Lecture Notes in Computer Science **1643**, 325–338 (1999). https://doi.org/10.1007/3-540-48481-7_29
16. Gao, S., Wang, C.: A new algorithm for the smallest enclosing circle. In: Proceedings of the 2018 8th International Conference on Management, Education and Information (MEICI 2018). pp. 562–567. Atlantis Press (2018/12). https://doi.org/10.2991/meici-18.2018.111
17. Goaoc, X., Welzl, E.: Convex hulls of random order types. Leibniz International Proceedings in Informatics, LIPIcs **164** (2020). https://doi.org/10.4230/LIPIcs.SoCG.2020.49
18. Har-Peled, S., Mazumdar, S.: Fast algorithms for computing the smallest k-enclosing circle. Algorithmica (New York) **41**(3), 147–157 (2005). https://doi.org/10.1007/s00453-004-1123-0
19. Jiang, Y., Cai, Y.: A reformulation-linearization based algorithm for the smallest enclosing circle problem. Journal of Industrial and Management Optimization **17**(6), 3633–3644 (2021). https://doi.org/10.3934/jimo.2020136
20. Jiang, Y., Luo, C., Ling, S.: An efficient cutting plane algorithm for the smallest enclosing circle problem. Journal of Industrial and Management Optimization **13**(1), 147–153 (2017). https://doi.org/10.3934/jimo.2016009
21. Kallberg, L., Shellshear, E., Larsson, T.: An external memory algorithm for the minimum enclosing ball problem. VISIGRAPP 2016 - Proceedings of the 11th Joint Conference on Computer Vision, Imaging and Computer Graphics Theory and Applications pp. 83–90 (2016). https://doi.org/10.5220/0005675600810088
22. Karmakar, A., Roy, S., Das, S.: Fast computation of smallest enclosing circle with center on a query line segment. CCCG 2007 - 19th Canadian Conference on Computational Geometry pp. 273–276 (2007)
23. Karmakar, A., Roy, S., Das, S.: Fast computation of smallest enclosing circle with center on a query line segment. Information Processing Letters **108**(6), 343–346 (2008). https://doi.org/10.1016/j.ipl.2008.07.002
24. Krivosija, A., Munteanu, A.: Probabilistic smallest enclosing ball in high dimensions via subgradient sampling. Leibniz International Proceedings in Informatics, LIPIcs **129** (2019). https://doi.org/10.4230/LIPIcs.SoCG.2019.47
25. Larsson, T., Kallberg, L.: Fast and robust approximation of smallest enclosing balls in arbitrary dimensions. Computer Graphics Forum **32**(5), 93–101 (2013). https://doi.org/10.1111/cgf.12176
26. Li, X., Ercan, M.: An algorithm for smallest enclosing circle problem of planar point sets. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics) **9786**, 309–318 (2016). https://doi.org/10.1007/978-3-319-42085-1_24
27. Liu, Y.F., Diao, R., Ye, F., Liu, H.W.: An efficient inexact Newton-CG algorithm for the smallest enclosing ball problem of large dimensions. Journal of the Operations Research Society of China **4**(2), 167–191 (2016). https://doi.org/10.1007/s40305-015-0097-8

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

28. Martinetz, T., Mamlouk, A., Mota, C.: Fast and easy computation of approximate smallest enclosing balls. Brazilian Symposium of Computer Graphic and Image Processing pp. 163–168 (2006). https://doi.org/10.1109/SIBGRAPI.2006.20
29. Martyn, T.: Tight bounding ball for affine IFS attractor. Computers and Graphics (Pergamon) **27**(4), 535–552 (2003). https://doi.org/10.1016/S0097-8493(03)00089-X
30. Matoušek, J., Sharir, M., Welzl, E.: A subexponential bound for linear programming. Algorithmica (New York) **16**(4-5), 498–516 (1996). https://doi.org/10.1007/bf01940877, the code available at https://news.ycombinator.com/item?id=14475832
31. Megiddo, N.: On the ball spanned by balls. Discrete & Computational Geometry **4**(1), 605–610 (1989). https://doi.org/10.1007/BF02187750
32. Mordukhovich, B., Nam, N., Villalobos, C.: The smallest enclosing ball problem and the smallest intersecting ball problem: Existence and uniqueness of solutions. Optimization Letters **7**(5), 839–853 (2013). https://doi.org/10.1007/s11590-012-0483-7
33. Mukherjee, D.: Reduction of two-dimensional data for speeding up convex hull computation (2022). https://doi.org/arxiv:2201.11412, `https://arxiv.org/pdf/2201.11412.pdf`
34. Munteanu, A., Sohler, C., Feldman, D.: Smallest enclosing ball for probabilistic data. Proceedings of the Annual Symposium on Computational Geometry pp. 214–223 (2014). https://doi.org/10.1145/2582112.2582114
35. Nam, N., Nguyen, T., Salinas, J.: Applications of convex analysis to the smallest intersecting ball problem. Journal of Convex Analysis **19**(2), 497–518 (2012)
36. Nielsen, F.: The siegel-klein disk: Hilbert geometry of the siegel disk domain. Entropy **22**(9) (2020). https://doi.org/10.3390/e22091019
37. Nielsen, F., Hadjeres, G.: Approximating covering and minimum enclosing balls in hyperbolic geometry. Lecture Notes in Computer Science **9389**, 586–594 (2015). https://doi.org/10.1007/978-3-319-25040-3_63
38. Nielsen, F., Nock, R.: Approximating smallest enclosing balls. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics) **3045**, 147–157 (2004). https://doi.org/10.1007/978-3-540-24767-8_16
39. Nielsen, F., Nock, R.: A fast deterministic smallest enclosing disk approximation algorithm. Information Processing Letters **93**(6), 263–268 (2005). https://doi.org/10.1016/j.ipl.2004.12.006
40. Nielsen, F., Nock, R.: On approximating the smallest enclosing bregman balls. Proceedings of the Annual Symposium on Computational Geometry **2006**, 485–486 (2006). https://doi.org/10.1145/1137856.1137931
41. Nielsen, F., Nock, R.: On the smallest enclosing information disk. Information Processing Letters **105**(3), 93–97 (2008). https://doi.org/10.1016/j.ipl.2007.08.007
42. Nielsen, F., Nock, R.: Approximating smallest enclosing balls with applications to machine learning. International Journal of Computational Geometry and Applications **19**(5), 389–414 (2009). https://doi.org/10.1142/S0218195909003039
43. Nock, R., Nielsen, P.: Fitting the smallest enclosing bregman ball. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics) **3720 LNAI**, 649–656 (2005). https://doi.org/10.1007/11564096_65
44. Pan, S., Li, X.: An efficient algorithm for the smallest enclosing ball problem in high dimensions. Applied Mathematics and Computation **172**(1), 49–61 (2006). https://doi.org/10.1016/j.amc.2005.01.127

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

45. Pronzato, L.: On the elimination of inessential points in the smallest enclosing ball problem. Optimization Methods and Software **34**(2), 225–247 (2019). https://doi.org/10.1080/10556788.2017.1359266
46. Ritter, J.: An efficient bounding sphere. Graphics Gems, Academic Press Professional, Inc. p. 301–303 (1990)
47. Saha, A., Vishwanathan, S., Zhang, X.: New approximation algorithms for minimum enclosing convex shapes. Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms pp. 1146–1160 (2011). https://doi.org/10.1137/1.9781611973082.86
48. Shen, K.W., Wang, X.K., Wang, J.Q.: Multi-criteria decision-making method based on smallest enclosing circle in incompletely reliable information environment. Computers and Industrial Engineering **130**, 1–13 (2019). https://doi.org/10.1016/j.cie.2019.02.011
49. Shenmaier, V.: Complexity and approximation of the smallest k-enclosing ball problem. European Journal of Combinatorics **48**, 81–87 (2015). https://doi.org/10.1016/j.ejc.2015.02.011
50. Shi, Y.Z., Wang, S.T., Wang, J., Deng, Z.H.: Fast classification for nonstationary large scale data sets using minimal enclosing ball. Kongzhi yu Juece/Control and Decision **28**(7), 1065–1072 (2013)
51. Skala, V.: Trading time for space: An O(1) average time algorithm for point-in-polygon location problem: Theoretical fiction or practical usage? Machine Graphics and Vision **5**(3), 483–494 (1996)
52. Skala, V.: Barycentric coordinates computation in homogeneous coordinates. Computers and Graphics (Pergamon) **32**(1), 120–127 (2008). https://doi.org/10.1016/j.cag.2007.09.007
53. Skala, V.: Fast $o_{expected}(n)$ algorithm for finding exact maximum distance in E2 instead of $O(N^2)$ or $O(N\ lgN)$. AIP Conference Proceedings **1558**, 2496–2499 (2013). https://doi.org/10.1063/1.4826047
54. Skala, V.: A new robust algorithm for computation of a triangle circumscribed sphere in E3 and a hypersphere simplex. AIP Conference Proceedings **1738** (2016). https://doi.org/10.1063/1.4952269
55. Skala, V.: Diameter and convex hull of points using space subdivision in E2 and E3. LNCS **12249**, 286–295 (2020). https://doi.org/10.1007/978-3-030-58799-4_21
56. Skala, V., Majdisova, Z.: Fast algorithm for finding maximum distance with space subdivision in E2. LNCS **9218**, 261–274 (2015). https://doi.org/10.1007/978-3-319-21963-9_24
57. Skala, V., Majdisova, Z., Smolik, M.: Space subdivision to speed-up convex hull construction in E3. Advances in Engineering Software **91**, 12–22 (2016). https://doi.org/10.1016/j.advengsoft.2015.09.002
58. Skala, V., Smolik, M.: Simple and fast $O_{exp}(N)$ algorithm for finding an exact maximum distance in E2 instead of $O(N^2)$ or $O(N \lg N)$. LNCS **11619**, 367–380 (2019). https://doi.org/10.1007/978-3-030-24289-3_27
59. Skala, V., Smolik, M., Majdisova, Z.: Reducing the number of points on the convex hull calculation using the polar space subdivision in E2. SIBGRAPI 2016 pp. 40–47 (2017). https://doi.org/10.1109/SIBGRAPI.2016.015
60. Skyum, S.: A simple algorithm for computing the smallest enclosing circle. Information Processing Letters **37**(3), 121–125 (1991). https://doi.org/10.1016/0020-0190(91)90030-L
61. Smolik, Z.M., Skala, V.: Efficient speed-up of the smallest enclosing circle algorithm. Informatica pp. 1–11 (2022). https://doi.org/10.15388/22-INFOR477, accepted for publication, online 2022-03-23

Computational Science, ICCS2022 conference, London, U.K., pp.720-733, LNCS 13350, Part I, Springer Nature, Switzerland, 2022
DOI: 10.1007/978-3-031-08751-6_52

62. Sylvester, J.: A question in the geometry of situation. Quarterly Journal of Pure and Applied Mathematics **1**, 79 (1857). https://doi.org/10.1049/iet-ipr.2016.0692
63. Tao, J.W., Wang, S.T.: Large margin and minimal reduced enclosing ball learning machine. Ruan Jian Xue Bao/Journal of Software **23**(6), 1458–1471 (2012). https://doi.org/10.3724/SP.J.1001.2012.04071
64. Wang, Y., Li, Y., Chang, L.: Approximate minimum enclosing ball algorithm with smaller core sets for binary support vector machine. 2010 Chinese Control and Decision Conference, CCDC 2010 pp. 3404–3408 (2010). https://doi.org/10.1109/CCDC.2010.5498584
65. Wang, Y., Li, Y., Tan, K.L.: Coresets for minimum enclosing balls over sliding windows. Proceedings of the ACM SIGKDD International Conference on Knowledge Discovery and Data Mining pp. 314–323 (2019). https://doi.org/10.1145/3292500.3330826
66. Wei, L.Y., Anand, A., Kumar, S., Beri, T.: Simple methods to represent shapes with sample spheres. SIGGRAPH Asia 2020 Technical Communications, SA 2020 (2020). https://doi.org/10.1145/3410700.3425424
67. Welzl, E.: Smallest enclosing disks (balls and ellipsoids). LNCS **555**, 359–370 (1991). https://doi.org/10.1007/BFb0038202
68. Welzl, E.: Geometric optimization and unique sink orientations of cubes. Lecture Notes in Computer Science **3153**, 176 (2004). https://doi.org/10.1007/978-3-540-28629-5_9
69. Welzl, E.: The smallest enclosing circle - a contribution to democracy from switzerland? Algorithms Unplugged pp. 357–360 (2011). https://doi.org/10.1007/978-3-642-15328-0_36
70. Wikipedia contributors: Article: Smallest-circle problem - Wikipedia, the free encyclopedia (2021), `https://en.wikipedia.org/wiki/Smallest-circle_problem`, [Online; accessed 29-January-2022]
71. Wikipedia contributors: Halton sequence - Wikipedia, the free encyclopedia (2021), `en.wikipedia.org/wiki/Halton_sequence`, [Online; accessed 27-January-2022]
72. Wikipedia contributors: Talk: Smallest-circle problem - Wikipedia, the free encyclopedia (2021), `https://en.wikipedia.org/wiki/Talk:Smallest-circle_problem`, [Online; accessed 29-January-2022]
73. Xu, J., Bu, F., Si, W., Qiu, Y., Chen, Z.: An algorithm of weighted Monte Carlo localization based on smallest enclosing circle. Proceedings - 2011 IEEE Int. Confl on Internet of Things and Cyber, Physical and Social Computing, iThings/CPSCom 2011 pp. 157–161 (2011). https://doi.org/10.1109/iThings/CPSCom.2011.67
74. Xu, S., Freund, R., Sun, J.: Solution methodologies for the smallest enclosing circle problem. Computational Optimization and Applications **25**(1-3), 283–292 (2003). https://doi.org/10.1023/A:1022977709811
75. Yildirim, E.: Two algorithms for the minimum enclosing ball problem. SIAM Journal on Optimization **19**(3), 1368–1391 (2008). https://doi.org/10.1137/070690419
76. Zhou, G., Tohemail, K.C., Sun, J.: Efficient algorithms for the smallest enclosing ball problem. Computational Optimization and Applications **30**(2), 147–160 (2005). https://doi.org/10.1007/s10589-005-4565-7
77. Zhou, Q., Zhu, H.S., Xu, Y.J., Li, X.W.: Smallest enclosing circle based localization approach for wireless sensor networks. Tongxin Xuebao/Journal on Communications **29**(11), 84–90 (2008)
78. Zhou, Y., Yang, B., Wang, J., Zhu, J., Tian, G.: A scaling-free minimum enclosing ball method to detect differentially expressed genes for rna-seq data. BMC genomics **22**(1), 479 (2021). https://doi.org/10.1186/s12864-021-07790-0