Radial Basis Functions Interpolation

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Outline

Introduction

Data collected in modern military operations are increasingly distributed in space and time. Examples include sensor arrays deployed over a battlefield, radar cross-section measurements, atmospheric sampling for UAV navigation, and underwater acoustic sensing.

However, such data are rarely structured: sensors move, fail, or report asynchronously.

Classical grid-based interpolation methods (finite differences, splines, or kriging) require regular grids or specific correlation models. In contrast, *Radial Basis Function* (RBF) methods operate directly on scattered data, requiring only pairwise distances.

This makes the RBF ideal for **spatial–temporal fusion of irregular samples**.

Introduction

Typical examples:

- **UAV Sensor Fusion**: Consider a surveillance scenario with UAVs sampling temperature or radiation intensity at scattered positions. The goal is to reconstruct the field $f(\mathbf{x})$ over the area for situational awareness.
- **Gap Filling:** Estimating values in regions where no measurements were taken (e.g., due to sensor failure, terrain obstruction, or enemy jamming).
- **Temporal Forecasting with RBFs**: For spatio–temporal tracking (e.g., chemical plume propagation or moving targets), the time dimension can be used for short-term prediction by extrapolating coefficients or fitting dynamic models on the reconstructed field.

Applied Military Examples:

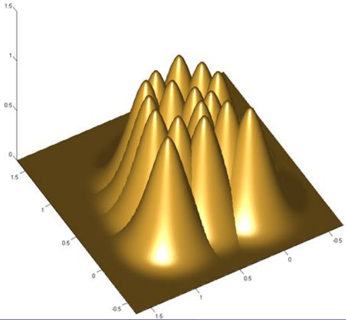
- **Battlefield Sensor Fusion**: Deployed ground sensors measure chemical concentrations. Due to communication limits, data arrive sporadically. RBF interpolation reconstructs the continuous concentration field in real time, aiding decision making.
- **Radar Cross-Section Mapping**: RBFs can interpolate measured radar reflections over frequency and angle. This provides smooth RCS maps for target identification and stealth evaluation.
- **UAV Swarm Coordination**: Each drone estimates wind speed at its position. RBF interpolation of these data produces a continuous vector field, enabling distributed path planning and collision avoidance.
- **Ocean and Atmospheric Models**: Underwater sensors record salinity, pressure, or acoustic intensity at irregular depths. RBF-based spatio-temporal interpolation reconstructs continuous environmental maps for navigation and communication systems.

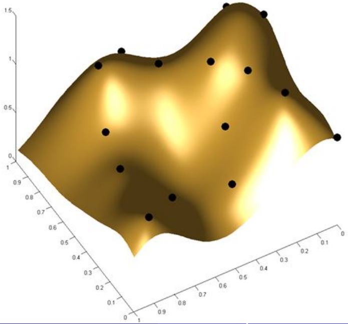
RBF Interpolation Principle

The key idea is to approximate a function $f(\mathbf{x})$ as a linear combination of basis kernel functions:

$$f(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \qquad (1)$$

where: $\phi_j(\mathbf{x}) = \phi(\|\mathbf{x} - \mathbf{x}_j\|)$ is the kernel RBF centered at $\mathbf{x}_j$, the coefficients c_j are to be determined and $\| * \|$ is a "distance" of the points $\mathbf{x}$ and $\mathbf{x}_j$.
RBFs are generally independent of the data domain dimension, and lead to a system of linear equations $\mathbf{Ax} = \mathbf{b}$.





RBF Kernel Functions

Kernel interpolation functions $\phi(r)$ are generally of two types:

- "**global**" functions with unlimited influence over the data domain, e.g. $\phi(r) = \frac{1}{2} r^2 \log r^2$, $\phi(r) = e^{-\alpha r^2}$, $\phi(r) = \sqrt{1+\alpha r^2}$, $\phi(r) = 1/\sqrt{1+\alpha r^2}$, The matrix $\mathbf{A}$ is usually "full" and might be ill-conditioned,
- "**local**" functions with a limited influence only for $r \in [0,1]$, mostly in the form
  $\phi(r) = (1-r)^q P_k(r)$, e.g. $(1-r)^5(8r^2+5r+1)_+$ "$_+$" means limitation to the interval $[0,1]$. A shape parameter $\alpha > 0$ is a multiplicative factor to r used to extend/shrink the function influence to $[0,\alpha]$.
  The matrix $\mathbf{A}$ is usually "sparse" (depends on α).
  or
- "**combined**": e.g. Buhmann's $\phi(r) = (\frac{1}{3} + r^2 - \frac{4}{3} r^3 + 2r^2 \log r)_+$ etc. or Menandro's functions: Two new classes of compactly supported radial basis functions for approximation of discrete and continuous data,
  DOI: 10.1002/eng2.12028

RBF Kernel Functions

Table: Common Types of Radial Basis Functions

Category	**Name**	**Definition** $\phi(r)$	**Parameters**
Global	Gaussian (GA)	$\exp(-(\epsilon r)^2)$	$\epsilon > 0$ (shape)
	Multiquadric (MQ)	$\sqrt{1+(\epsilon r)^2}$	$\epsilon > 0$
	Inverse Multiquadric (IMQ)	$1/\sqrt{1+(\epsilon r)^2}$	$\epsilon > 0$
	Thin Plate Spline (TPS)	$r^2 \log(r)$	(none)
Compact	Wendland C^2	$(1-\epsilon r)_+^4(4\epsilon r+1)$	$\epsilon > 0,\ r \leq 1/\epsilon$

Note: The subscript '$+$' denotes the cutoff function: $(x)_+ = x$ if $x \geq 0$, and 0 otherwise.

Scalar value interpolation

If scalar values $h_i = h(\mathbf{x}_i)$ are given in mutually distinct scattered points $\mathbf{x}_i$, $i = 1, \dots, N$ a system of linear equations is obtained:

$$h(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x}_i - \mathbf{x}_j\|) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi_{ij} \quad , \quad i = 1, \dots, N \tag{2}$$

where: $\phi(\|\mathbf{x} - \mathbf{x}_j\|) = \phi_j(\mathbf{x})$ is the kernel RBF centered at $\mathbf{x}_j$, c_j are coefficients to be determined, $\mathbf{x}_i$ are the interpolation points, $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ is the Euclidean distance, $\phi_j(\mathbf{x}_i) = \phi_{ij}$ is the value of the function ϕ_j at the point $\mathbf{x}_i$.
This formulation leads to a system of linear equations $\mathbf{Ax} = \mathbf{b}$, i.e.

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix} \tag{3}$$

where the matrix $\mathbf{A}$ is symmetrical, i.e. $\mathbf{A} = \mathbf{A}^T$.

Scalar value interpolation

To obtain better numerical robustness and matrix positivity, a polynomial $P_k(\mathbf{x})$ of the degree k is usually added together with additional orthogonal conditions, e.g., for the degree $k = 2$ the polynomial can have a form $P_2(\mathbf{x}) = a_0 + a_1 x + a_2 y + a_3 xy$. Then, the interpolation is given as:

$$h(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x}_i - \mathbf{x}_j\|) + P_k(\mathbf{x}_i) \ , \ \ i = 1, \dots, N$$

$$P_k(\mathbf{x}) = a_0 + a_1 x + a_2 y + a_3 xy \qquad (4)$$

$$\sum_{j=1}^{N} a_0 = 0 \ , \ \sum_{j=1}^{N} a_1 x_j = 0 \ , \ \sum_{j=1}^{N} a_2 y_j = 0 \ , \ \sum_{j=1}^{N} a_3 x_j y_j = 0$$

It should be noted, that the size of the matrix $\mathbf{A}$ is independent of the data domain dimension.

Scalar value interpolation

This leads to the linear system of equations:

$$\left[\begin{array}{cccc|cccc} \phi_{11} & \phi_{12} & \dots & \phi_{1N} & 1 & x_1 & y_1 & x_1 y_1 \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} & 1 & x_2 & y_2 & x_2 y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} & 1 & x_N & y_N & x_N y_N \\ \hline 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & \dots & x_N & 0 & 0 & 0 & 0 \\ y_1 & y_2 & \dots & y_N & 0 & 0 & 0 & 0 \\ x_1 y_1 & x_2 y_2 & \dots & x_N y_N & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_N \\ \hline a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right] = \left[\begin{array}{c} h_1 \\ h_2 \\ \vdots \\ h_N \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \quad (5)$$

It can be rewritten as:

$$\begin{bmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{h} \\ 0 \end{bmatrix} \quad (6)$$

where $\mathbf{a} = [a_0, \dots, a_3]^T$, $\mathbf{c} = [c_1, \dots, c_N]^T$, $\mathbf{h} = [h_1, \dots, h_N]^T$ and $h_i = f(\mathbf{x}_i)$, $i = 1, \dots, N$.

Generally, the polynomial $P_k(\mathbf{x})$ can be seen as a "global approximation" and RBF as a "local adjustment of values". It also leads to better numerical robustness and interpolation precision.

Scalar value interpolation CSRBF

It can be seen that the matrix in Eq.6 can be quite large and dense if global RBFs are used. However, if Compactly Supported RBFs (CS-RBF) are used, and the shape parameter α is adequately chosen the matrix of the linear system is sparse. It leads to faster computation of large data sets, and data domain subdivisions can be used.

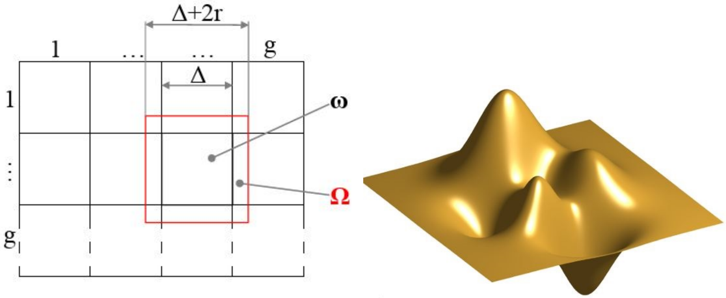


Figure: Data domain subdivision and blended interpolation [14]

Scalar value interpolation

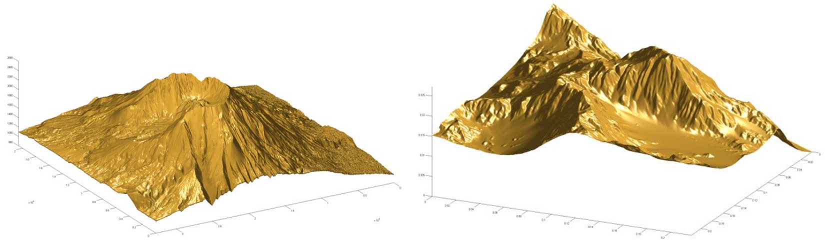


Figure: Mount St. Helens; 756 150 points   Alps mountains; 131 044 points [14]

RBF interpolation on larger datasets using Compactly Supported RBF(CS-RBF) and domain space subdivision was described in:

- Michal Smolik and Vaclav Skala. “Large scattered data interpolation with radial basis functions and space subdivision”. In: *Integrated Computer-Aided Engineering* 25.1 (2017), pp. 49–62. ISSN: 1069-2509. DOI: 10.3233/ICA-170556 [14], IF=5.8
- Vaclav Skala. “RBF Interpolation with CSRBF of Large Data Sets”. In: *Procedia Computer Science* 108 (2017). ICCS 2017,Zurich, Switzerland, pp. 2433–2437. ISSN: 1877-0509. DOI: 10.1016/j.procs.2017.05.081 [12]

It leads to significant computational speedup, lower memory demands, and better numerical stability. In the case of the St.Helens, the RFB coefficients computation speedup was $\nu \approx 2.4\ 10^5$, and RBF evaluation speedup was $\nu \approx\ 172$.
The existence of *individual* optimal shape parameters for each CS-RBF kernel function:

- Vaclav Skala, Samsul Ariffin Abdul Karim, and Marek Zabran. “Radial basis function approximation optimal shape parameters estimation”. In: *Lecture Notes in Computer Science* 12142 LNCS (2020), pp. 309–317. ISSN: 0302-9743. DOI: 10.1007/978-3-030-50433-5_24 [2]

Scalar value approximation

In the case of over-sampled data or large data sets, approximation can be used as RBF interpolation leads to large ill-conditioned matrices, e.g., in the case of the Mount St. Helens, the interpolation matrix was approx. $6.7\ 10^6 \times 6.7\ 10^6$ [9].

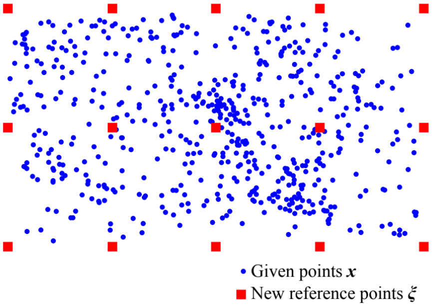


Figure: The reference points (knots) can be distributed arbitrarily[9].

The main idea in RBF approximation is to select M reference points (knots) $\boldsymbol{\xi}_j$, $j = 1, \ldots, M$, where the precision of the data approximation is critical; the reference points must be carefully chosen [5, 4].

Scalar value approximation

The RBF approximation for N given values using M knots is formulated as:

$$h(\mathbf{x}_i) = \sum_{j=1}^{M} c_j \phi(\|\mathbf{x}_i - \boldsymbol{\xi}_j\|) = \sum_{j=1}^{M} c_j \phi_j(\mathbf{x}_i) = \sum_{j=1}^{M} c_j \phi_{ij}, \;\; i = 1, \ldots, N, \;\; M \ll N \quad (7)$$

where: $\phi(\|\mathbf{x} - \boldsymbol{\xi}_j\|) = \phi_j(\mathbf{x})$ is the kernel RBF centered at $\boldsymbol{\xi}_j$, c_j are coefficients to be determined, $\mathbf{x}_i$ are the interpolation points, $r_{ij} = \|\mathbf{x}_i - \boldsymbol{\xi}_j\|$ is the Euclidean distance between a given point $\mathbf{x}_i$ and a reference point $\boldsymbol{\xi}_j$.
This formulation leads to an over-determined system of linear equations $\mathbf{Ax} = \mathbf{b}$:

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1M} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NM} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_M \end{bmatrix} \quad (8)$$

Scalar value approximation

An additional polynomial $P_k(\mathbf{x})$ of the degree k can be used as well:

$$h(\mathbf{x}_i) = \sum_{j=1}^{M} c_j \phi(\|\mathbf{x}_i - \boldsymbol{\xi}_j\|) + P_k(\mathbf{x}_i), \;\; i = 1, \ldots, N, \;\; M \ll N \tag{9}$$

The over-determined system of linear equations, Eq.9, has the form:

$$\left[\begin{array}{cccc|cccc} \phi_{11} & \phi_{12} & \ldots & \phi_{1M} & 1 & x_1 & y_1 & x_1 y_1 \\ \phi_{21} & \phi_{22} & \ldots & \phi_{2M} & 1 & x_2 & y_2 & x_2 y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \ldots & \phi_{NM} & 1 & x_N & y_N & x_N y_N \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_M \\ \hline a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right] = \left[\begin{array}{c} h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_N \end{array}\right] \tag{10}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{h} \end{bmatrix} \tag{11}$$

However, in this case, pseudo-inverse, or the Least Square Error, cannot be used directly[1] [9]; the Lagrange multipliers have to be used instead [10].

Scalar value approximation

If the CS-RBFs are used, even a huge linear system can be solved using the block matrix decomposition and specialized data structures for sparse matrices.

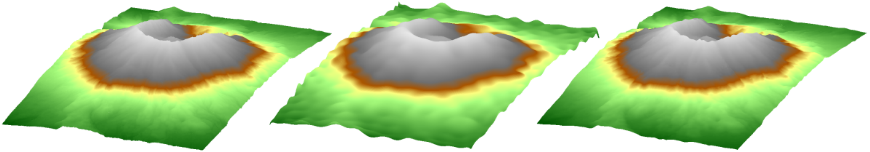


Figure: St.Helen Mount: Original, Gauss RBF $\alpha = 4\ 10^{-4}$, Wenland's $\phi_{3,1}$, $\alpha = 0.01$ [16]

- Zuzana Majdisova and Vaclav Skala. "Radial basis function approximations: comparison and applications". In: *Applied Mathematical Modelling* 51 (2017), pp. 728–743. ISSN: 0307-904X. DOI: 10.1016/j.apm.2017.07.033 [10], IF=4.4
- Zuzana Majdisova and Vaclav Skala. "Big geo data surface approximation using radial basis functions: A comparative study". In: *Computers and Geosciences* 109 (2017), pp. 51–58. ISSN: 0098-3004. DOI: 10.1016/j.cageo.2017.08.007 [9], IF=4.2
- Zuzana Majdisova, Vaclav Skala, and Michal Smolik. "Algorithm for placement of reference points and choice of an appropriate variable shape parameter for the RBF approximation". In: *Integrated Computer-Aided Engineering* 27.1 (2019), pp. 1–15. ISSN: 1069-2509. DOI: 10.3233/ICA-190610 [5], IF=5.8
- Martin Cervenka, Michal Smolik, and Vaclav Skala. "A New Strategy for Scattered Data Approximation Using Radial Basis Functions Respecting Points of Inflection". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 322–336. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_24 [4]
- Vaclav Skala. "RBF Approximation of Big Data Sets with Large Span of Data". In: *MCSI 2017 Proceedings* 2018-January (2017), pp. 212–218. DOI: 10.1109/MCSI.2017.44 [11]

The Gauss RBF is often used for PDE solutions; it generally leads to very ill-conditioned matrices with high sensitivity to the shape parameter α value.

Vector Fields approximation

Approximation of vector data is an often task, especially in the case of acquired data. The vector field data are usually extensive and describe complex physical behavior. It means that the reference points for approximation must be chosen carefully, and critical points[2] detection, classification[13], and evaluation[15] are to be used.

Critical points of vector field are defined as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) = \mathbf{0} \tag{12}$$

Usually, only linearized vector field $\mathbf{v}(t)$ and simple classification of critical points are used, based on Jacobian eigenvalues. see Fig.7.

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x_0}, t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x_0}) = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{bmatrix} (\mathbf{x} - \mathbf{x_0}) = \mathbf{J}(\mathbf{x} - \mathbf{x_0}) \tag{13}$$

where $\mathbf{f}(\mathbf{x_0}, t) = \mathbf{0}$ in the critical point [8].

Vector Fields approximation

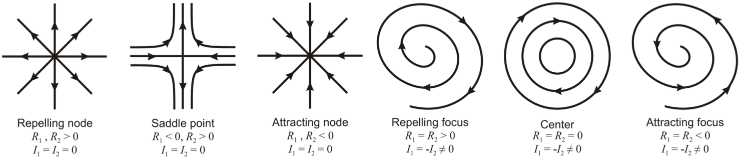


Figure: Non-repealing nodes Repealing node // Critical points classification [8]

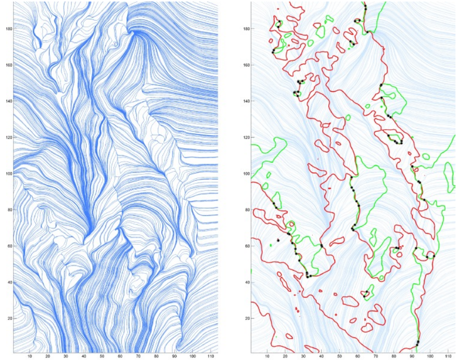
The RBFs can be used for vector data approximation [8] with a high compression ratio using critical points classification, see Fig.7.

- Michal Smolik, Vaclav Skala, and Zuzana Majdisova. "Vector field radial basis function approximation". In: *Advances in Engineering Software* 123 (2018), pp. 117–129. ISSN: 0965-9978. DOI: 10.1016/j.advengsoft.2018.06.013 [8], IF=4.0

Also, the level of details approach can be used efficiently [1] to reduce vector field data for visualization.

- Michal Smolik and Vaclav Skala. "Radial basis function and multi-level 2D vector field approximation". In: *Mathematics and Computers in Simulation* 181 (2021), pp. 522–538. ISSN: 0378-4754. DOI: 10.1016/j.matcom.2020.10.009 [1], IF=4.4

Vector Fields approximation



(a) Vector field. (b) Zero iso-lines and critical points.

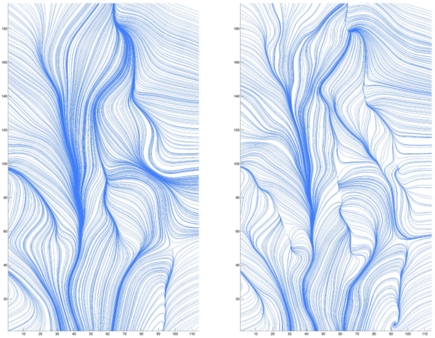


Figure: Approximation&compression [8]   Different levels of details [1]

In the vector field classification, the first-degree derivatives are usually used.

Vector Fields approximation

The Hessian matrix can be used in a critical point analysis.

$$v_x = \mathbf{J}_x(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}_x(\mathbf{x}-\mathbf{x}_0),\ v_y = \mathbf{J}_y(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}_y(\mathbf{x}-\mathbf{x}_0)$$

where $\mathbf{H}_x$ and $\mathbf{H}_y$ are Hessian matrices and $\mathbf{J}_x$, resp. $\mathbf{J}_y$ is the first, resp. the second row of the Jacobian matrix $\mathbf{J}$.

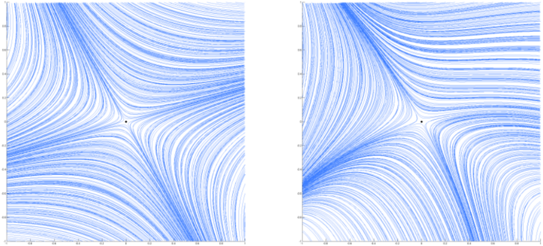


Figure: Critical points with linear approximation and second order derivatives influence [15].

The influences of the second-order derivatives were published in:

- Michal Smolik and Vaclav Skala. "Classification of Critical Points Using a Second Order Derivative". In: *Procedia Computer Science* 108 (2017). ICCS 2017, Zurich, Switzerland, pp. 2373–2377. ISSN: 1877-0509. DOI: 10.1016/j.procs.2017.05.271 [13]
- Michal Smolik and Vaclav Skala. "Vector field second order derivative approximation and geometrical characteristics". In: *Lecture Notes in Computer Science* 10404 (2017), pp. 148–158. ISSN: 0302-9743. DOI: 10.1007/978-3-319-62392-4_11 [15]

Vector Fields approximation

Besides the critical "linearized" critical points and the influence of second derivatives in the critical points, the influence of inflection points using second derivatives was studied. A vector field can be given in the explicit or implicit form.

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \quad \text{or} \quad F(\mathbf{x}, t) = 0 \tag{14}$$

where $\mathbf{f}(\mathbf{x}, t) = [f_x(\mathbf{x}, t), f_y(\mathbf{x}, t)]^T$. Then

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{dt} + \frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial t} \tag{15}$$

If autonomous ODEs are considered, i.e. $\partial \mathbf{F}(\mathbf{x}, t)/\partial t = 0$ then Eq.15 is simplified:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{F}(\mathbf{x}, t)}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{dt} = \nabla F \; \dot{\mathbf{x}} \tag{16}$$

Then the extreme and inflection points are given as:

$$\det \mathbf{Q}(x, y) = \begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{yx} & F_{xx} & F_y \\ F_x & F_y & 0 \end{vmatrix} = 0 \tag{17}$$

Vector Fields approximation

This enables us to select additional important points (knots) more efficiently.

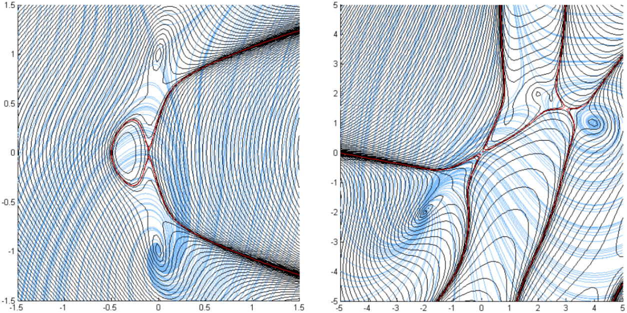


Figure: Vector field with two and three critical points [6].

Research results were also published in:

- Vaclav Skala and Michal Smolik. "A new approach to vector field interpolation, classification and robust critical points detection using radial basis functions". In: *Advances in Intelligent Systems and Computing* 765 (2019), pp. 109–115. ISSN: 2194-5357. DOI: 10.1007/978-3-319-91192-2_12 [6]
- Martin Cervenka, Michal Smolik, and Vaclav Skala. "A New Strategy for Scattered Data Approximation Using Radial Basis Functions Respecting Points of Inflection". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 322–336. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_24 [4]

Vector Fields approximation

Application of CS-RBF approximation in 3D using space subdivision to 3D tornado data set containing approx. 5.5 10^8 3D points led to a significant speed-up of interpolation computation and also to a high compression rate (7 10^3 : 1) with high approximation precision [7], see Fig.10.

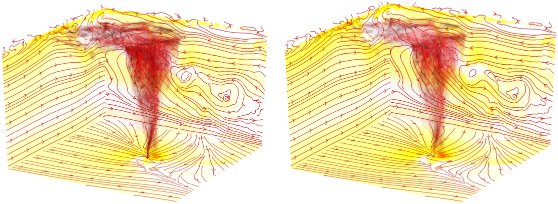


Figure: Tornado 3D original data   Tornado 3D approximated data
5.5 10^8 points approx. error 0.1%, compression rate 1 : 10^3
RBF Tornado data approximation [7]
The data set of EF5 tornado courtesy of Leigh Orf from Cooperative Institute for Meteorological Satellite Studies, University of Wisconsin, Madison, WI, USA

- Michal Smolik and Vaclav Skala. "Efficient Simple Large Scattered 3D Vector Fields Radial Basis Functions Approximation Using Space Subdivision". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 337–350. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_25 [7]

Summary - Main advantages

- **Simplicity**: The formulation leads to a simple formula relying on a solution of linear systems of equations, which is heavily supported by numerical libraries with efficient solution methods.
- **No meshing (tessellation) required**: The method relies only on the mutual positions of scattered points in the domain, making it particularly useful for irregular domains or problems with moving boundaries.
- **Flexibility**: RBFs can be chosen based on the problem, with typical choices including Gaussian, multi-quadric, thin-plate splines (TPS), or Wenland's functions.
- **Smoothness**: RBF interpolation typically results in a smooth and accurate approximation; however, the interpolation error might be higher on the data domain border.
- **Independence on data domain dimensionality**: RBF use leads to a system of linear equations $\mathbf{Ax} = \mathbf{b}$, where the matrix $\mathbf{A}$ is of size $N \times N$, but is independent of the dimensionality of the data domain.
- **Speed-up**: If Compactly Supported RBFs (CS-RBFs) are used, the matrix $\mathbf{A}$ can be sparse and specific methods can be used.
- **Space subdivision**: If the CS-RBFs are used, space subdivision of the data domain can be used to significantly speed up the solution and the function $f(\mathbf{x})$ evaluation.

Summary - Main disadvantages

- **Numerical problems**: The system of linear equations can be very large for large data sets.
  In [16] over 6.7 mil. of points have been interpolated, i.e., the RBF matrix was over the size $6.7\ 10^6 \times 6.7\ 10^6$ was processed.
  Zuzana Majdisova and Vaclav Skala. "A Radial Basis Function Approximation for Large Datasets". In: *SIGRAD 2016*. http://www.ep.liu.se/ecp/127/002/ecp16127002.pdf. Visby, Sweden, 2016, pp. 9–14 [16]
- **Numerical robustness**: The matrix **A** tends to be very ill-conditioned, primarily when "global" kernel functions are used.
- **Time consuming**: Evaluation of the interpolated value $h(\mathbf{x})$ can be unacceptable for high N and if the matrix **A** is dense.
- **Parameter value**: Some kernel RBFs are sensitive to choosing a "shape" parameter α value with influence on RBF matrix conditionality.

It should be noted that the RBF, which is used with a polynomial, cannot be directly used for approximation using the least square errors method.

- Zuzana Majdisova and Vaclav Skala. "Big geo data surface approximation using radial basis functions: A comparative study". In: *Computers and Geosciences* 109 (2017), pp. 51–58. ISSN: 0098-3004. DOI: 10.1016/j.cageo.2017.08.007 [9]

Applications

RBF-based meshless methods are widely used in fields such as interpolation, approximation, computational fluid dynamics, structural mechanics, image processing, GIS systems, and data fitting, especially when mesh generation is challenging or computationally expensive and/or higher dimension of the data domain.

Military Applications

Terrain Reconstruction and Elevation Modeling

UAVs and reconnaissance patrols often collect scattered elevation points. RBFs can create a smooth, continuous Digital Elevation Model (DEM):

- Line-of-sight analysis for sniper placement or communication relay siting.
- Path planning for autonomous ground vehicles.
- Mission planning and simulation.

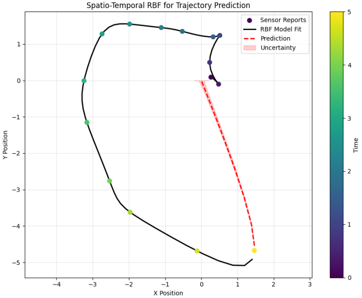


Figure: Spatio-temporal RBF interpolation for trajectory prediction. Scattered position reports (blue dots) are used to fit a spatio-temporal RBF model. The model can be used to predict the future path (red dashed line) and estimate uncertainty (gray cone).

Military Applications

The TPS RBF is particularly well-suited for this task as it minimizes the bending energy of the surface, creating a physically plausible "smoothest" surface.

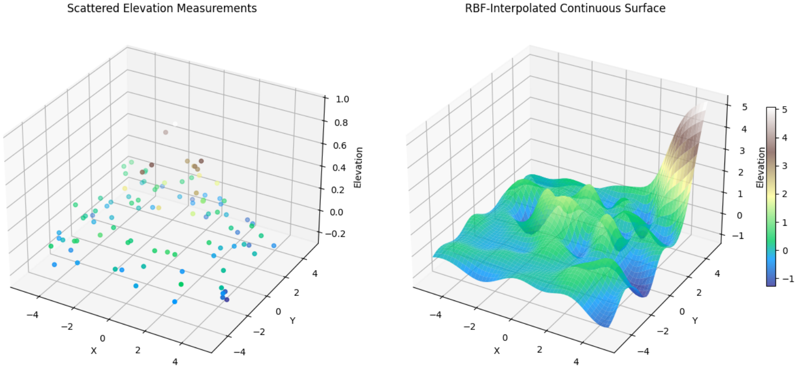


Figure: Reconstruction of a terrain surface from scattered elevation points using RBF interpolation.

Military Applications

Sensor Fusion for Comprehensive Air Picture
Multiple radars (e.g., long-range early warning and short-range point defence) provide overlapping but incomplete tracks on multiple targets. The data is noisy, asynchronous, and may contain false tracks. RBFs can be used to:

- **Fuse Tracks:** Create a single, coherent, and accurate estimate of each target's position and velocity.
- **De-ghosting:** Resolve ambiguities that arise when multiple radars see the same target from different angles, potentially creating "ghost" targets.

A spatio-temporal RBF model can assimilate all position reports in space and time, creating a continuous track for each target that is more accurate and reliable than any single sensor's output.

Military Applications

Prediction of Threat Trajectories As shown conceptually, RBFs are excellent for predicting the future path of dynamic threats.

By modeling the target's history as a spatio-temporal curve, the RBF interpolant can be extrapolated to provide a probabilistic forecast. This is vital for:

- **Missile Defence:** Predicting the impact point of an incoming ballistic missile.
- **Counter-UAS:** Forecasting the flight path of a hostile drone to enable effective interdiction.

Conclusions

- RBF interpolation and approximation provide an elegant and flexible method for reconstructing smooth fields from irregularly distributed spatial–temporal data.
- Mesh-free nature, local adaptability, and robustness to missing data make them highly suitable for defence and security systems.
- Applications range from UAV mapping and radar analysis, radiation level estimation, submarine position detection to environmental reconstruction and real-time sensor fusion, etc.
- Sensors report radiation and chemical agent concentration at various (x, y) locations and times. Using this approach, we can reconstruct the continuous 4D concentration field $C(x, y, z, t)$ and predict its future dispersion.
- Trajectory Prediction: Forecasting the future position of aircraft, missiles, or ground vehicles.
- Threat Anticipation: Predicting the likely path of a spreading forest fire or flood towards a base of operations.

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Geographical distribution of RBF citations

Figure: Citation geographical distribution of all publications

Thanks your for your attention!

Questions?