Meshless Methods in Computer Graphics and Visualization An Overview

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2. RBF Meshless Methods - Principle
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4. RBF Approximation
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Introduction

Meshless methods are used in:

- interpolation and approximation of d-dimensional **scattered spatio-temporal** data,
- solution of differential equations (ODE & PDE - Kansa's method),
- surface reconstruction, shape deformation & geometry processing,
- surface fitting - implicit RBFs (Carr's implicit RBF, Velho's Hermite RBF),
- image processing - image interpolation, scattered image resampling, super-resolution, denoising, feature interpolation,
- scientific computing & visualization,
- signal processing, sensor field modeling, RBF-NN, machine learning

No data domain tessellation is needed (triangulation, tetrahedronization etc.)

- Fasshauer,G.E.: "Meshfree approximation methods with MATLAB" (2007), DOI: 10.1142/6437

RBF Interpolation Principle

The key idea is to approximate a function $f(\mathbf{x})$ as a linear combination of basis kernel functions:

$$f(\mathbf{x}) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \tag{1}$$

where: $\phi_j(\mathbf{x}) = \phi(\|\mathbf{x} - \mathbf{x}_j\|)$ is the kernel RBF centered at $\mathbf{x}_j$, the coefficients c_j are to be determined and $\| * \|$ is a "distance" of the points $\mathbf{x}$ and $\mathbf{x}_j$.
RBFs are generally independent of the data domain dimension, and lead to a linear system $\mathbf{Ax} = \mathbf{b}$.

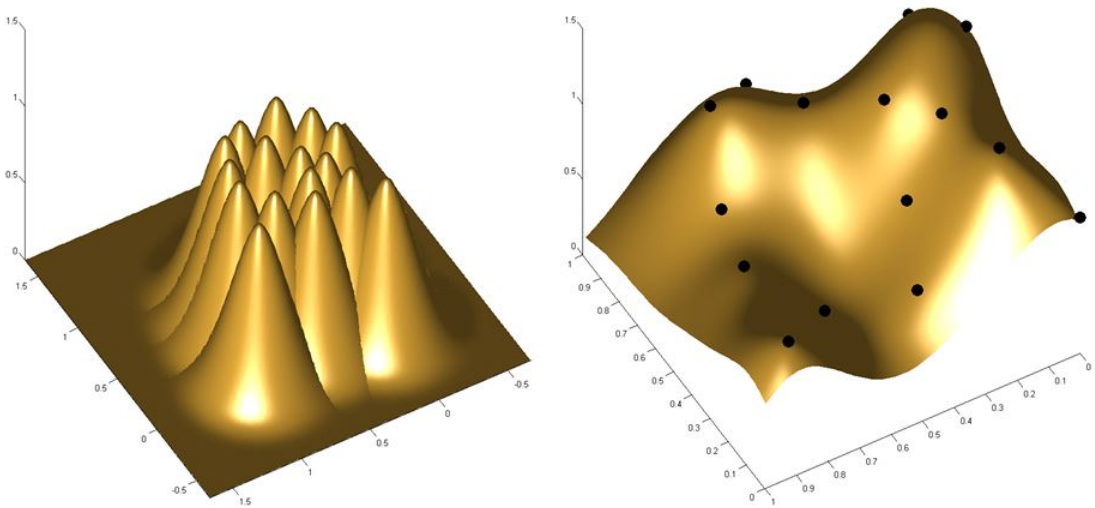


Figure: Kernel functions and final RBF interpolation [4]

RBF Kernel Functions

Kernel interpolation functions $\phi(r)$ are generally of two types:

- "**global**" functions with unlimited influence over the data domain,
  e.g. $\phi(r) = \frac{1}{2} r^2 \log r^2$, $\phi(r) = e^{-\alpha r^2}$, $\phi(r) = \sqrt{1 + \alpha r^2}$, $\phi(r) = 1/\sqrt{1 + \alpha r^2}$,
  The matrix **A** is usually "full" and might be ill-conditioned,
- "**local**" functions with a limited influence only for $r \in [0, 1]$, mostly in the form
  $\phi(r) = (1 - r)^q P_k(r)$, e.g. $(1 - r)^5 (8r^2 + 5r + 1)_+$ "$_+$" means limitation to the interval $[0, 1]$. A shape parameter $\alpha > 0$ is a multiplicative factor to r used to extend/shrink the function influence to $[0, \alpha]$.
  The matrix **A** is usually "sparse" (depends on α).
- "combined"
  Buhmann's $\phi(r) = (\frac{1}{3} + r^2 - \frac{4}{3} r^3 + 2r^2 \log r)_+$, Menandro's, etc.
  - Menandro,F.C.: Two new classes of compactly supported radial basis functions for approximation of discrete and continuous data, DOI: 10.1002/eng2.12028

Scalar value interpolation

If scalar values $h_i = h(\mathbf{x}_i)$ are given in mutually distinct scattered points $\mathbf{x}_i$, $i = 1, \ldots, N$ a system of linear equations is obtained:

$$h(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x}_i - \mathbf{x}_j\|) = \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi_{ij} \quad , \quad i = 1, \ldots, N \tag{2}$$

where: $\phi(\|\mathbf{x} - \mathbf{x}_j\|) = \phi_j(\mathbf{x})$ is the kernel RBF centered at $\mathbf{x}_j$, c_j are coefficients to be determined, $\mathbf{x}_i$ are the interpolation points, $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ is the Euclidean distance, $\phi_j(\mathbf{x}_i) = \phi_{ij}$ is the value of the function ϕ_j at the point $\mathbf{x}_i$.
This formulation leads to a system of linear equations $\mathbf{Ax} = \mathbf{b}$, i.e.

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \ldots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \ldots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \ldots & \phi_{NN} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix} \tag{3}$$

where the matrix $\mathbf{A}$ is symmetrical, i.e. $\mathbf{A} = \mathbf{A}^T$.

Scalar value interpolation

To obtain better numerical robustness and matrix positivity, a polynomial $P_k(\mathbf{x})$ of degree k is usually added together with additional orthogonal conditions, e.g., for the degree $k = 2$, the polynomial can have a form $P_2(\mathbf{x}) = a_0 + a_1 x + a_2 y + a_3 xy$. The interpolation is given as:

$$h(\mathbf{x}_i) = \sum_{j=1}^{N} c_j \phi(\|\mathbf{x}_i - \mathbf{x}_j\|) + P_k(\mathbf{x}_i) \ , \ i = 1, \ldots, N$$

$$P_k(\mathbf{x}) = a_0 + a_1 x + a_2 y + a_3 xy \tag{4}$$

$$\sum_{j=1}^{N} c_j = 0 \ , \ \sum_{j=1}^{N} c_j x_j = 0 \ , \ \sum_{j=1}^{N} c_j y_j = 0 \ , \ \sum_{j=1}^{N} c_j x_j y_j = 0$$

- Size of the matrix **A** is *independent of the dimension of the data domain*
- Numerical problems in the large span data case
- Invariance loss (translation, rotation)

Scalar value interpolation

It leads to the linear system of equations:

$$\left[\begin{array}{cccc|cccc} \phi_{11} & \phi_{12} & \dots & \phi_{1N} & 1 & x_1 & y_1 & x_1y_1 \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} & 1 & x_2 & y_2 & x_2y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} & 1 & x_N & y_N & x_Ny_N \\ \hline 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 \\ x_1 & x_2 & \dots & x_N & 0 & 0 & 0 & 0 \\ y_1 & y_2 & \dots & y_N & 0 & 0 & 0 & 0 \\ x_1y_1 & x_2y_2 & \dots & x_Ny_N & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_N \\ \hline a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right] = \left[\begin{array}{c} h_1 \\ h_2 \\ \vdots \\ h_N \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{5}$$

It can be rewritten as:

$$\begin{bmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{h} \\ 0 \end{bmatrix} \tag{6}$$

where $\mathbf{a} = [a_0, \dots, a_3]^T$, $\mathbf{c} = [c_1, \dots, c_N]^T$, $\mathbf{h} = [h_1, \dots, h_N]^T$ and $h_i = f(\mathbf{x}_i)$.
The polynomial $P_k(\mathbf{x})$ can be seen as a "global approximation" [low frequency] and RBF's $\phi(*)$ as a "local adjustment of values" [high frequency].
It also leads to better numerical robustness and interpolation precision.
The matrix $\mathbf{A}$ can be quite large and dense if global RBFs are used.

Scalar value interpolation CS-RBF

If **Compactly Supported** RBFs (CS-RBF) are used, and the shape parameter α is "adequately" chosen the matrix of the linear system is sparse.
It leads to faster computation of large data sets.
Also, data domain subdivisions can be used in the CS-RBF case.

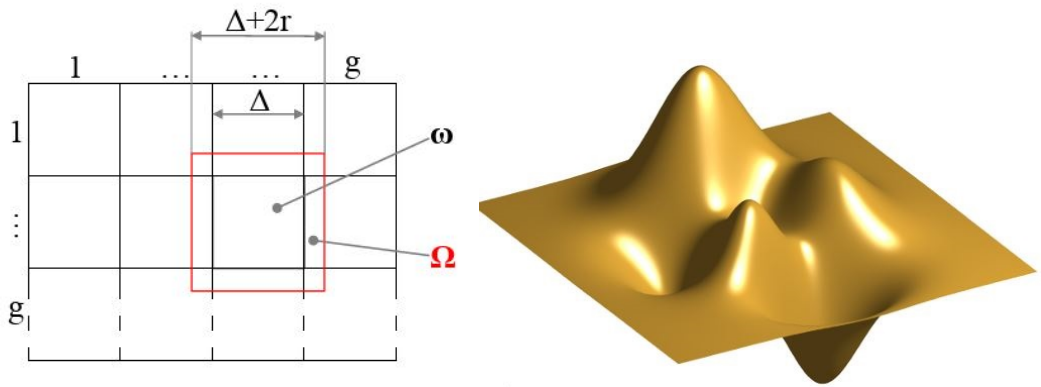


Figure: Data domain subdivision and blended interpolation [15]

- Michal Smolik and Vaclav Skala. "Large scattered data interpolation with radial basis functions and space subdivision". In: *Integrated Computer-Aided Engineering* 25.1 (2017), pp. 49–62. ISSN: 1069-2509. DOI: 10.3233/ICA-170556 [15] IF=5.8

Scalar value interpolation

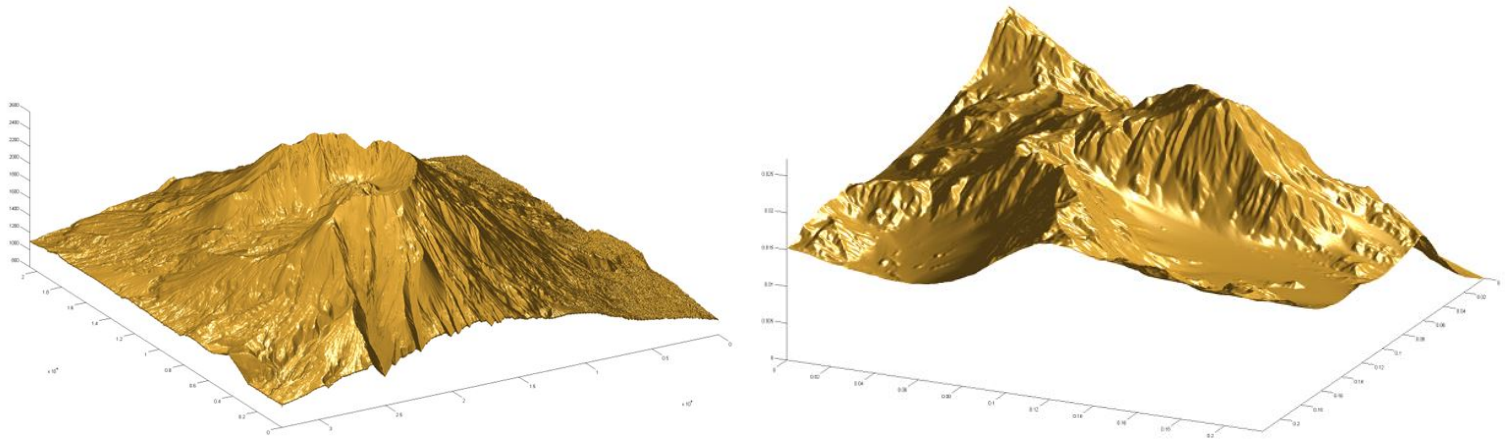


Figure: St. Helens; 756 150 points (approximation) Alps mountains; 131 044 points [15]

RBF on larger datasets using CS-RBF and domain space subdivision was described in:

- Michal Smolik and Vaclav Skala. "Large scattered data interpolation with radial basis functions and space subdivision". In: *Integrated Computer-Aided Engineering* 25.1 (2017), pp. 49–62. ISSN: 1069-2509. DOI: 10.3233/ICA-170556 [15], IF=5.8
- Vaclav Skala. "RBF Interpolation with CSRBF of Large Data Sets". In: *Procedia Computer Science* 108 (2017). ICCS 2017,Zurich, Switzerland, pp. 2433–2437. ISSN: 1877-0509. DOI: 10.1016/j.procs.2017.05.081 [13]

CS-RBF leads to significant computational speedup, lower memory demands, and better numerical stability.
In the case of the St.Helens, the RBF coefficients computation speedup was $\nu \approx 2.4\ 10^5$, and RBF function $f(\mathbf{x})$ evaluation speedup was $\nu \approx 172$.
The existence of *individual* optimal shape parameters for each CS-RBF kernel function:

- Vaclav Skala, Samsul Ariffin Abdul Karim, and Marek Zabran. "Radial basis function approximation optimal shape parameters estimation". In: *Lecture Notes in Computer Science* 12142 LNCS (2020), pp. 309–317. ISSN: 0302-9743. DOI: 10.1007/978-3-030-50433-5_24 [3]
  Detailed tests: http://wscg.zcu.cz/RBF-shape/contents-new.htm

Scalar value interpolation - polynomial coefficients

RBF interpolation is given as:

$$\begin{bmatrix} \mathbf{A} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{h} \\ 0 \end{bmatrix} \tag{7}$$

Where **c** represents "high frequency" and **a** represents "low frequency" behavior.

Questions:

- Can the RBF polynomial coefficients be expressed explicitly in an analytical form?
- What is the meaning of the a_0 coefficient?
  (an average, or a mean of **h** values?)

Scalar value interpolation

The coefficients $\mathbf{a}$ of the k-degree polynomial $P_k(\mathbf{x})$ are given* as [1]:

$$\mathbf{a} = \left(\mathbf{P}^T\mathbf{A}^{-1}\mathbf{P}\right)^{-1}\mathbf{P}^T\mathbf{A}^{-1}\mathbf{h} \tag{8}$$

If a zero-degree polynomial is used the a_0 coefficient is given as:

$$a_0 = \left(\mathbf{1}^T\mathbf{A}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}^T\mathbf{A}^{-1}\mathbf{h} = \frac{\mathbf{1}^T\mathbf{A}^{-1}\mathbf{h}}{\mathbf{1}^T\mathbf{A}^{-1}\mathbf{1}} \tag{9}$$

i.e.

$$\boxed{a_0 = \frac{\sum_{i,j=1}^{n}(\mathbf{A}^{-1})_{ij}h_j}{\sum_{i,j=1}^{n}(\mathbf{A}^{-1})_{ij}} = \frac{\sum_{j=1}^{n}h_j\sum_{i=1}^{n}\alpha_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{n}\alpha_{ij}}} \tag{10}$$

where α_{ij} are elements of the inverse matrix $\mathbf{A}^{-1}$.

- *Vaclav Skala. "Analytical formulas for polynomial coefficients in radial basis function interpolation". In: *Mathematical Modelling and Analysis* 31.1 (Jan. 2026), pp. 130–148. ISSN: 1392-6292. DOI: 10.3846/mma.2026.22788 [1], IF=1.4 (2024), Q1

Scalar value approximation

In the case of over-sampled data or large data sets, approximation can be used as RBF interpolation leads to large ill-conditioned matrices, e.g., in the case of Mount St. Helens, the **RBF interpolation matrix** was approx. $6.7\ 10^6 \times 6.7\ 10^6$ [10].

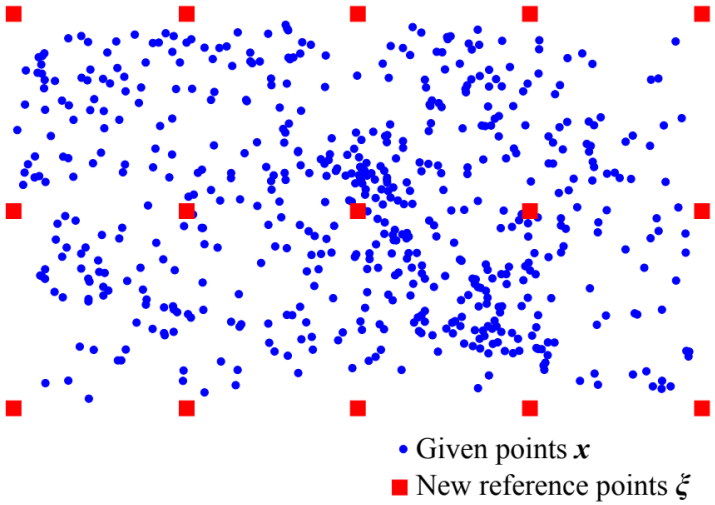


Figure: The reference points (knots) can be distributed arbitrarily[10].

The main idea in RBF approximation is to select M reference points (knots) $\boldsymbol{\xi}_j$, $j = 1, \ldots, M$, where the precision of the data approximation is critical; the reference points must be carefully chosen [5, 6].

- Zuzana Majdisova, Vaclav Skala, and Michal Smolik. "Algorithm for placement of reference points and choice of an appropriate variable shape parameter for the RBF approximation". In: *Integrated Computer-Aided Engineering* 27.1 (2019), pp. 1–15. ISSN: 1069-2509. DOI: 10.3233/ICA-190610 [6] IF=5.8
- Zuzana Majdisova and Vaclav Skala. "A Radial Basis Function Approximation for Large Datasets". In: *SIGRAD 2016*. http://www.ep.liu.se/ecp/127/002/ecp16127002.pdf. Visby, Sweden, 2016, pp. 9–14 [17] (dblp indexed)

Scalar value approximation

The RBF approximation for N given values using M knots is formulated as:

$$h(\mathbf{x}_i) = \sum_{j=1}^{M} c_j \phi(\|\mathbf{x}_i - \boldsymbol{\xi}_j\|) = \sum_{j=1}^{M} c_j \phi_j(\mathbf{x}_i) = \sum_{j=1}^{M} c_j \phi_{ij},\ i = 1, \ldots, N,\ M \ll N \quad (11)$$

where: $\phi(\|\mathbf{x} - \boldsymbol{\xi}_j\|) = \phi_j(\mathbf{x})$ is the kernel RBF centered at $\boldsymbol{\xi}_j$, c_j are coefficients to be determined, $\mathbf{x}_i$ are the interpolation points, $r_{ij} = \|\mathbf{x}_i - \boldsymbol{\xi}_j\|$ is the Euclidean distance between a given point $\mathbf{x}_i$ and a reference point $\boldsymbol{\xi}_j$.
This formulation leads to an over-determined system of linear equations $\mathbf{Ax} = \mathbf{b}$:

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1M} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \cdots & \phi_{NM} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_N \end{bmatrix} \quad (12)$$

Scalar value approximation

An additional polynomial $P_k(\mathbf{x})$ of the degree k can be used as well:

$$h(\mathbf{x}_i) = \sum_{j=1}^{M} c_j \phi(\|\mathbf{x}_i - \boldsymbol{\xi}_j\|) + P_k(\mathbf{x}_i),\ i = 1, \ldots, N,\ M \ll N \tag{13}$$

The over-determined system of linear equations, Eq.13, has the form:

$$\left[\begin{array}{cccc|cccc} \phi_{11} & \phi_{12} & \ldots & \phi_{1M} & 1 & x_1 & y_1 & x_1 y_1 \\ \phi_{21} & \phi_{22} & \ldots & \phi_{2M} & 1 & x_2 & y_2 & x_2 y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \ldots & \phi_{NM} & 1 & x_N & y_N & x_N y_N \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_M \\ \hline a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right] = \left[\begin{array}{c} h_1 \\ h_2 \\ \vdots \\ \vdots \\ h_N \end{array}\right] \tag{14}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{a} \end{bmatrix} = \begin{bmatrix} \mathbf{h} \end{bmatrix} \quad \text{, i.e.} \quad \mathbf{Ac} + \mathbf{Pa} = \mathbf{h} \tag{15}$$

The Lagrange multipliers have to be used [10]. The coefficients of the $P_k(\mathbf{x})$ can be expressed analytically[1].

- Vaclav Skala. "Analytical formulas for polynomial coefficients in radial basis function interpolation". In: *Mathematical Modelling and Analysis* 31.1 (Jan. 2026), pp. 130–148. ISSN: 1392-6292. DOI: 10.3846/mma.2026.22788 [1] IF=1.4 (2024), Q1
- Zuzana Majdisova and Vaclav Skala. "Big geo data surface approximation using radial basis functions: A comparative study". In: *Computers and Geosciences* 109 (2017), pp. 51–58. ISSN: 0098-3004. DOI: 10.1016/j.cageo.2017.08.007 [10] IF=2.721

Scalar value approximation

If CS-RBFs are used, a large linear system can be solved using block matrix decomposition and specialized data structures for sparse matrices.

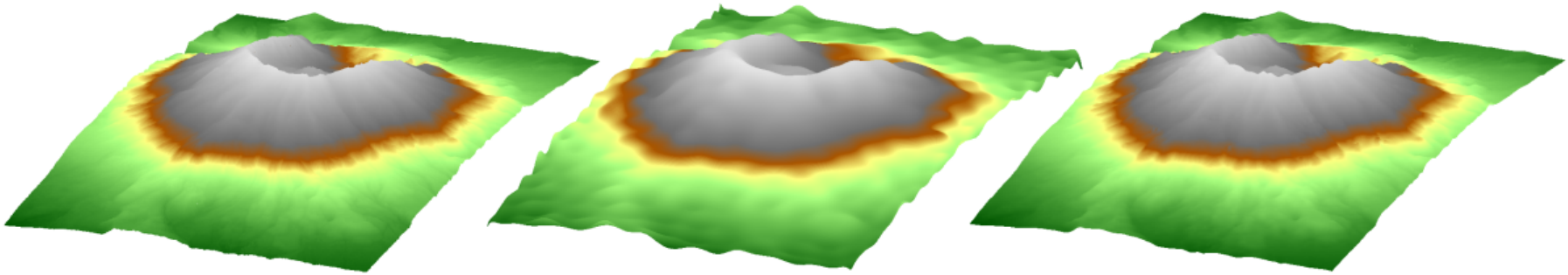


Figure: St.Helen Mount: Original, Gauss RBF $\alpha = 4\ 10^{-4}$, Wenland's $\phi_{3,1}$, $\alpha = 0.01$ [17]

- Zuzana Majdisova and Vaclav Skala. "Radial basis function approximations: comparison and applications". In: *Applied Mathematical Modelling* 51 (2017), pp. 728–743. ISSN: 0307-904X. DOI: 10.1016/j.apm.2017.07.033 [11], IF=4.4
- Zuzana Majdisova and Vaclav Skala. "Big geo data surface approximation using radial basis functions: A comparative study". In: *Computers and Geosciences* 109 (2017), pp. 51–58. ISSN: 0098-3004. DOI: 10.1016/j.cageo.2017.08.007 [10], IF=4.2
- Zuzana Majdisova, Vaclav Skala, and Michal Smolik. "Algorithm for placement of reference points and choice of an appropriate variable shape parameter for the RBF approximation". In: *Integrated Computer-Aided Engineering* 27.1 (2019), pp. 1–15. ISSN: 1069-2509. DOI: 10.3233/ICA-190610 [6], IF=5.8
- Martin Cervenka, Michal Smolik, and Vaclav Skala. "A New Strategy for Scattered Data Approximation Using Radial Basis Functions Respecting Points of Inflection". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 322–336. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_24 [5]
- Vaclav Skala. "RBF Approximation of Big Data Sets with Large Span of Data". In: *MCSI 2017 Proceedings* 2018-January (2017), pp. 212–218. DOI: 10.1109/MCSI.2017.44 [12]

The Gauss RBF is often used for PDE solutions; it generally leads to very ill-conditioned matrices with high sensitivity to the shape parameter α value.

Vector Fields approximation

Approximation of **vector data** is a frequent task in the case of acquired data. The vector field data are usually extensive and describe complex physical behavior. The reference points* for approximation must be chosen carefully, and critical points detection, classification[14], and evaluation[16] are to be used.
Critical points of vector field are defined as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) = \mathbf{0} \tag{16}$$

In data visualization, a linearized vector field $\mathbf{v}(t)$ and a simple classification of critical points based on the evaluation of eigenvalues according to the Jacobian equation are usually used.

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x_0}, t) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x_0}) = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{bmatrix} (\mathbf{x} - \mathbf{x_0}) = \mathbf{J}(\mathbf{x} - \mathbf{x_0}) \tag{17}$$

where $\mathbf{f}(\mathbf{x_0}, t) = \mathbf{0}$ in the critical point [9].

*Critical points of a vector field are points where the magnitude of the vector is zero.

Vector Fields approximation

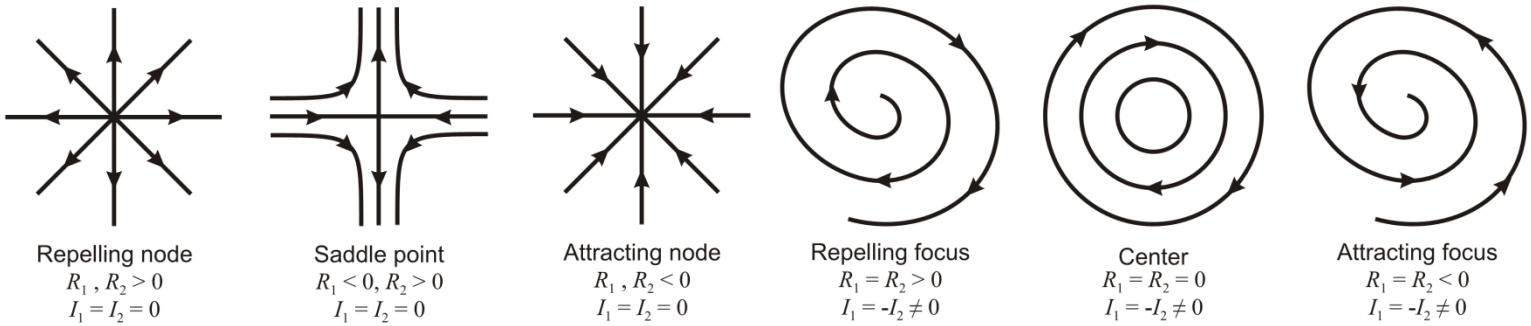


Figure: Non-repealing nodes Repealing node // Critical points classification [9]

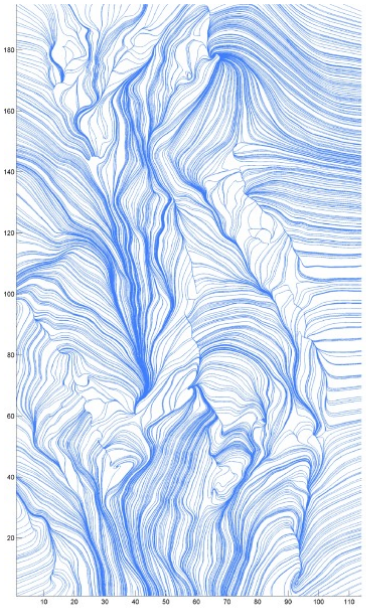
The RBFs can be used for vector data approximation [9] with a high compression ratio using critical points classification.

- Michal Smolik, Vaclav Skala, and Zuzana Majdisova. "Vector field radial basis function approximation". In: *Advances in Engineering Software* 123 (2018), pp. 117–129. ISSN: 0965-9978. DOI: 10.1016/j.advengsoft.2018.06.013 [9], IF=4.0

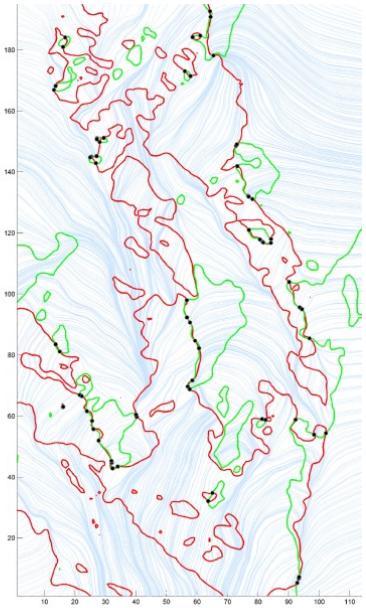
The level of detail approach can be used efficiently [8] to reduce vector field data for visualization.

- Michal Smolik and Vaclav Skala. "Radial basis function and multi-level 2D vector field approximation". In: *Mathematics and Computers in Simulation* 181 (2021), pp. 522–538. ISSN: 0378-4754. DOI: 10.1016/j.matcom.2020.10.009 [2], IF=4.4
- Michal Smolik and Vaclav Skala. "Efficient Simple Large Scattered 3D Vector Fields Radial Basis Functions Approximation Using Space Subdivision". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 337–350. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_25 [8]

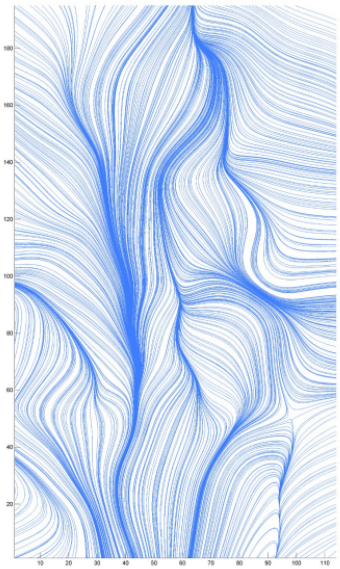
Vector Fields approximation



(a) Vector field.



(b) Zero iso-lines and critical points.



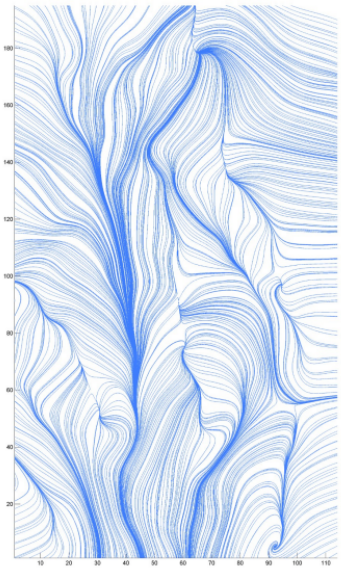


Figure: Approximation&compression [9] Different levels of details [2]

The first-degree derivatives are usually used in vector field classifications.

- Michal Smolik and Vaclav Skala. "Radial basis function and multi-level 2D vector field approximation". In: *Mathematics and Computers in Simulation* 181 (2021), pp. 522–538. ISSN: 0378-4754. DOI: 10.1016/j.matcom.2020.10.009 [2] IF=4.4, Q2/Q1 (Wos/Scopus)

Vector Fields approximation

The Hessian matrix can be used in a critical point analysis.

$$v_x = \mathbf{J}_x(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}_x(\mathbf{x}-\mathbf{x}_0), v_y = \mathbf{J}_y(\mathbf{x}-\mathbf{x}_0) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}_y(\mathbf{x}-\mathbf{x}_0)$$

where $\mathbf{H}_x$ and $\mathbf{H}_y$ are Hessian matrices and $\mathbf{J}_x$, resp. $\mathbf{J}_y$ is the first, resp. the second row of the Jacobian matrix $\mathbf{J}$.

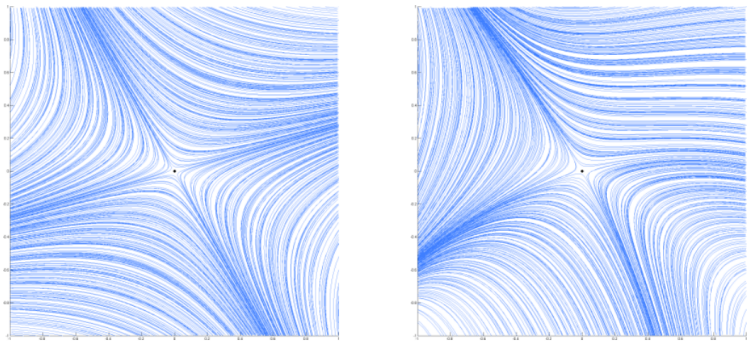


Figure: Critical points with linear approximation and second order derivatives influence [16].

The influences of the second-order derivatives were published in:

- Michal Smolik and Vaclav Skala. "Classification of Critical Points Using a Second Order Derivative". In: *Procedia Computer Science* 108 (2017). ICCS 2017, Zurich, Switzerland, pp. 2373–2377. ISSN: 1877-0509. DOI: 10.1016/j.procs.2017.05.271 [14]
- Michal Smolik and Vaclav Skala. "Vector field second order derivative approximation and geometrical characteristics". In: *Lecture Notes in Computer Science* 10404 (2017), pp. 148–158. ISSN: 0302-9743. DOI: 10.1007/978-3-319-62392-4_11 [16]

Vector Fields approximation

The influence of second derivatives at the critical points.
This enables us to select additional important points (knots) more efficiently.

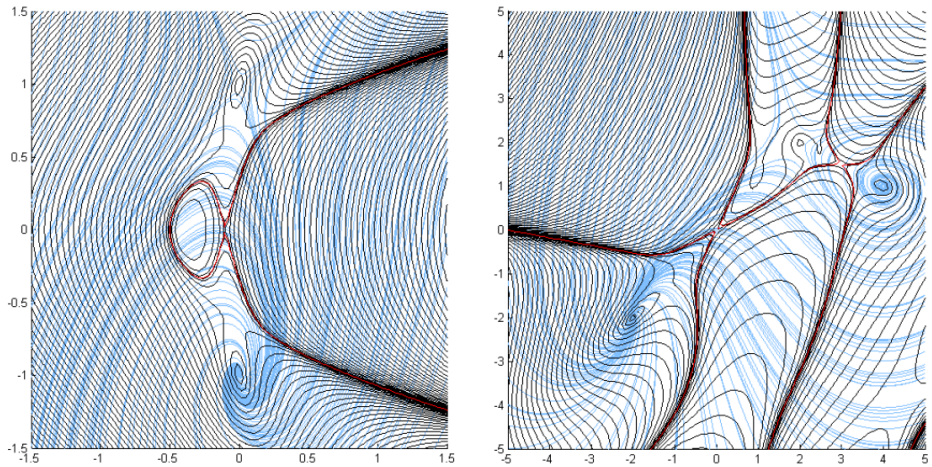


Figure: Vector field with two and three critical points [7].

Results were published in:

- Vaclav Skala and Michal Smolik. "A new approach to vector field interpolation, classification and robust critical points detection using radial basis functions". In: *Advances in Intelligent Systems and Computing* 765 (2019), pp. 109–115. ISSN: 2194-5357. DOI: 10.1007/978-3-319-91192-2_12 [7]
- Martin Cervenka, Michal Smolik, and Vaclav Skala. "A New Strategy for Scattered Data Approximation Using Radial Basis Functions Respecting Points of Inflection". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 322–336. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_24 [5]

Vector Fields approximation - Application

CS-RBF approximation of the 3D tornado data set containing approx. 5.5 10^8 of 3D points:

- significant speed-up of interpolation computation
- high compression rate (7 10^3 : 1) with high approximation precision [8].

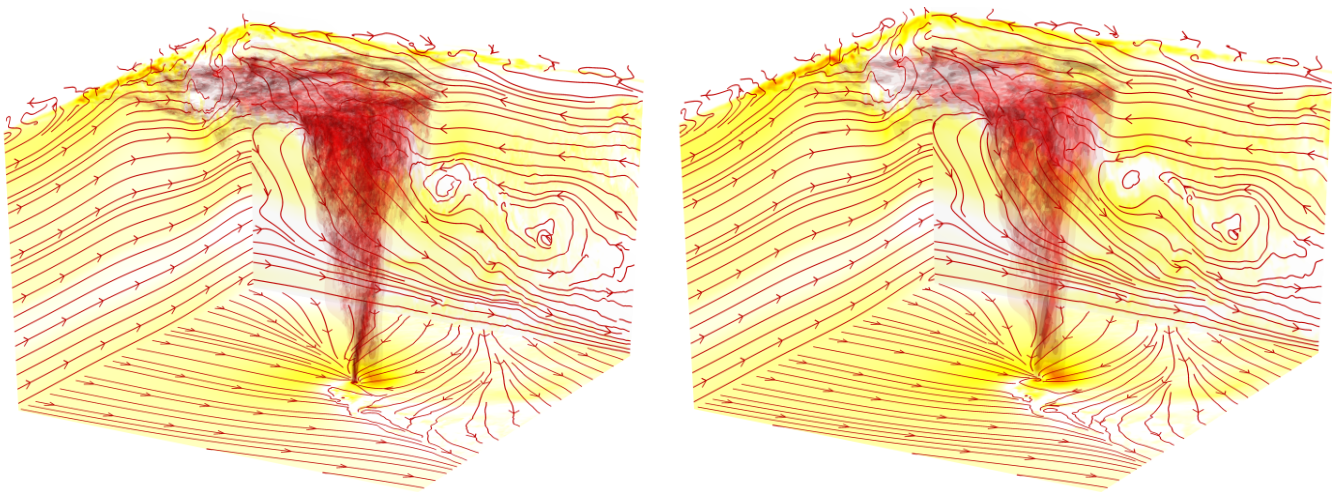


Figure: Tornado 3D original data 5.5 10^8 points Tornado 3D approximated data approx. error 0.1%, compression rate 1 : 10^3

- Michal Smolik and Vaclav Skala. "Efficient Simple Large Scattered 3D Vector Fields Radial Basis Functions Approximation Using Space Subdivision". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 337–350. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_25 [8]

The data set of EF5 tornado from Cooperative Institute for Meteorological Satellite Studies, University of Wisconsin, Madison, WI, USA.

Summary

Advantages

- **Simplicity**: The formulation leads to a simple formula relying on a solution of linear systems of equations, which is heavily supported by numerical libraries with efficient solution methods.
- **No meshing (tessellation) required**: The method relies only on the mutual positions of scattered points in the domain, making it particularly useful for irregular domains or problems with moving boundaries.
- **Flexibility**: RBFs can be chosen based on the problem, with typical choices including Gaussian, multi-quadric, thin-plate splines (TPS), or Wenland's functions.
- **Smoothness**: RBF interpolation typically results in a **smooth** and accurate approximation; however, the interpolation error might be higher on the data domain border.
- **Independence on data domain dimensionality**: RBF use leads to a system of linear equations $\mathbf{Ax} = \mathbf{b}$, where the matrix $\mathbf{A}$ is of size $N \times N$, but is independent of the dimensionality of the data domain.
- **Speed-up**: If Compactly Supported RBFs (CS-RBFs) are used, the matrix $\mathbf{A}$ can be sparse and specific methods can be used.
- **Space subdivision**: If the CS-RBFs are used, space subdivision of the data domain can be used to significantly speed up the solution and the function $f(\mathbf{x})$ evaluation.
- **Applicability**: RBF can be used for **scattered spatio-temporal data** interpolation/approximation in d-dimensional space, including **scattered in time**, i.e. "*non-framed in time*" data.

Summary

Main disadvantages

- **Numerical problems**: The system of linear equations can be very large for large data sets. In [17] over 6.7 mil. of points have been interpolated, i.e., the RBF matrix was over the size $6.7\ 10^6 \times 6.7\ 10^6$ was processed.
  - Zuzana Majdisova and Vaclav Skala. "A Radial Basis Function Approximation for Large Datasets". In: *SIGRAD 2016*. http://www.ep.liu.se/ecp/127/002/ecp16127002.pdf. Visby, Sweden, 2016, pp. 9–14 [17]
- **Numerical robustness**: The matrix **A** tends to be very ill-conditioned, primarily when "global" kernel functions are used.
- **Time consuming**: Evaluation of the interpolated value $h(\mathbf{x})$ can be unacceptable for high N and if the matrix **A** is dense, if not CS-RBF used.
- **Parameter value**: Some kernel RBFs are sensitive to choosing a "shape" parameter α value with influence on RBF matrix conditionality.

The least square errors method cannot be directly used if the RBF approximation with a polynomial due to mixture of physical units.

- Zuzana Majdisova and Vaclav Skala. "Big geo data surface approximation using radial basis functions: A comparative study". In: *Computers and Geosciences* 109 (2017), pp. 51–58. ISSN: 0098-3004. DOI: 10.1016/j.cageo.2017.08.007 [10]

Applications

RBF-based meshless methods are widely used in fields such as interpolation, approximation, computational fluid dynamics, structural mechanics, image processing, GIS systems, and data fitting, especially when mesh generation is challenging or computationally expensive and/or higher dimension of the data domain.

Acknowledgment

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Thanks also go to the participants of international tutorials, courses, and workshops on the RBF topic that I have led abroad.

REFERENCES

The most relevant selected publications:

[1] Vaclav Skala. "Analytical formulas for polynomial coefficients in radial basis function interpolation". In: *Mathematical Modelling and Analysis* 31.1 (Jan. 2026), pp. 130–148. ISSN: 1392-6292. DOI: 10.3846/mma.2026.22788.

[2] Michal Smolik and Vaclav Skala. "Radial basis function and multi-level 2D vector field approximation". In: *Mathematics and Computers in Simulation* 181 (2021), pp. 522–538. ISSN: 0378-4754. DOI: 10.1016/j.matcom.2020.10.009.

[3] Vaclav Skala, Samsul Ariffin Abdul Karim, and Marek Zabran. "Radial basis function approximation optimal shape parameters estimation". In: *Lecture Notes in Computer Science* 12142 LNCS (2020), pp. 309–317. ISSN: 0302-9743. DOI: 10.1007/978-3-030-50433-5_24.

[4] Michal Smolik and Vaclav Skala. "Efficient Speed-Up of Radial Basis Functions Approximation and Interpolation Formula Evaluation". In: *Lecture Notes in Computer Science* 12249 LNCS (2020), pp. 165–176. ISSN: 0302-9743. DOI: 10.1007/978-3-030-58799-4_12.

[5] Martin Cervenka, Michal Smolik, and Vaclav Skala. "A New Strategy for Scattered Data Approximation Using Radial Basis Functions Respecting Points of Inflection". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 322–336. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_24.

[6] Zuzana Majdisova, Vaclav Skala, and Michal Smolik. "Algorithm for placement of reference points and choice of an appropriate variable shape parameter for the RBF approximation". In: *Integrated Computer-Aided Engineering* 27.1 (2019), pp. 1–15. ISSN: 1069-2509. DOI: 10.3233/ICA-190610.

[7] Vaclav Skala and Michal Smolik. "A new approach to vector field interpolation, classification and robust critical points detection using radial basis functions". In: *Advances in Intelligent Systems and Computing* 765 (2019), pp. 109–115. ISSN: 2194-5357. DOI: 10.1007/978-3-319-91192-2_12.

[8] Michal Smolik and Vaclav Skala. "Efficient Simple Large Scattered 3D Vector Fields Radial Basis Functions Approximation Using Space Subdivision". In: *Lecture Notes in Computer Science* 11619 LNCS (2019), pp. 337–350. ISSN: 0302-9743. DOI: 10.1007/978-3-030-24289-3_25.

[9] Michal Smolik, Vaclav Skala, and Zuzana Majdisova. "Vector field radial basis function approximation". In: *Advances in Engineering Software* 123 (2018), pp. 117–129. ISSN: 0965-9978. DOI: 10.1016/j.advengsoft.2018.06.013.

[10] Zuzana Majdisova and Vaclav Skala. "Big geo data surface approximation using radial basis functions: A comparative study". In: *Computers and Geosciences* 109 (2017), pp. 51–58. ISSN: 0098-3004. DOI: 10.1016/j.cageo.2017.08.007.

[11] Zuzana Majdisova and Vaclav Skala. "Radial basis function approximations: comparison and applications". In: *Applied Mathematical Modelling* 51 (2017), pp. 728–743. ISSN: 0307-904X. DOI: 10.1016/j.apm.2017.07.033.

[12] Vaclav Skala. "RBF Approximation of Big Data Sets with Large Span of Data". In: *MCSI 2017 Proceedings* 2018-January (2017), pp. 212–218. DOI: 10.1109/MCSI.2017.44.

[13] Vaclav Skala. "RBF Interpolation with CSRBF of Large Data Sets". In: *Procedia Computer Science* 108 (2017). ICCS 2017,Zurich, Switzerland, pp. 2433–2437. ISSN: 1877-0509. DOI: 10.1016/j.procs.2017.05.081.

[14] Michal Smolik and Vaclav Skala. "Classification of Critical Points Using a Second Order Derivative". In: *Procedia Computer Science* 108 (2017). ICCS 2017, Zurich, Switzerland, pp. 2373–2377. ISSN: 1877-0509. DOI: 10.1016/j.procs.2017.05.271.

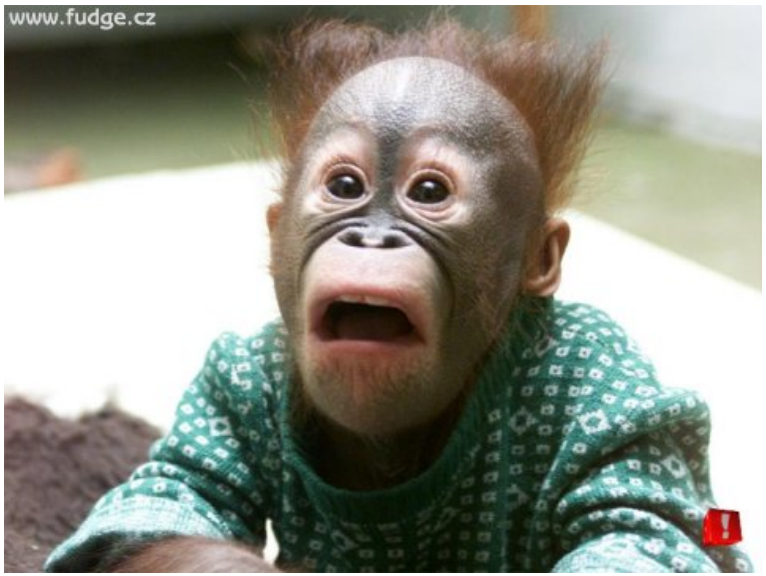
[15] Michal Smolik and Vaclav Skala. "Large scattered data interpolation with radial basis functions and space subdivision". In: *Integrated Computer-Aided Engineering* 25.1 (2017), pp. 49–62. ISSN: 1069-2509. DOI: 10.3233/ICA-170556.

[16] Michal Smolik and Vaclav Skala. "Vector field second order derivative approximation and geometrical characteristics". In: *Lecture Notes in Computer Science* 10404 (2017), pp. 148–158. ISSN: 0302-9743. DOI: 10.1007/978-3-319-62392-4_11.

[17] Zuzana Majdisova and Vaclav Skala. "A Radial Basis Function Approximation for Large Datasets". In: *SIGRAD 2016*. http://www.ep.liu.se/ecp/127/002/ecp16127002.pdf. Visby, Sweden, 2016, pp. 9–14.

Thank you for your attention

Questions



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